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A Consumption-Saving Model in the Infinite Time Horizon

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Abstract: This paper deals with the consumption-saving model in the infinite time horizon with the Epstein-Zin-Weil aggregator function. We formulate the dynamic recursive optimality equation, find the value function and the optimal policy of a consumer. We also provide some examples.

Key words: consumption-saving model, infinite time horizon, value function
JEL code: G

1. Introduction

Recursive utility models of Kreps and Porteus (1978) and models presented in the literature of asset valuation by Epstein and Zin (1989) and others represent investor preferences as a solution to a nonlinear, forward looking functional equation. Preferences such as those given in Becker R. A. & Boyd III H. J. (1997) and Miao J. (2014) are used in economic dynamics because they constitute a convenient tool for changing risk aversion while maintaining the same elasticity of intertemporal substitution (EIS). In this article, I deal with the consumption model considered in an infinite time horizon. We are considering a decision problem in which the investor has to decide at every step how much to consume and how much to invest in the future, see Stokey N., Lucas R., and Prescott E. (1989). We solve this problem by examining models with a finite horizon. We show that under a mild assumption, the sequence of value functions obtained for problems with a finite time horizon converges as the time horizon approaches infinity. It turns out that the limit is equal to the value function over an infinite time horizon. We prove that the value function satisfies the so-called optimality equation and we calculate the optimal policy.

2. The Model and Results

2.1 Elasticity of Intertemporal Substitution

Preferences over a consumption bundle $c = (c_t)_{t=1}^{\infty}$ at different points in time should be represented by a utility function of the form

$$U(c) = U(c_1, c_2, \dots)$$

If we put ${}_tc = (c_t, c_{t+1})$; then $U(c) = U({}_1c)$. Let us define

$$U_t = \frac{\partial U({}_tc)}{\partial c_t}$$

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Elasticity of intertemporal substitution (EIS) between consecutive dates t and $t + 1$ is an evaluation of

$$EIS = \left|\frac{dln(c_{t+1}/c_t)}{dln(U_{t+1}/U_t)}\right|$$

In microeconomics the quotient U_{t+1}/U_t is interpreted as the ratio of prices of consumption of one unit of a single good in periods $t + 1$ and t.

Note that

$$d\,ln(c_{t+1}/c_t) = \frac{d(c_{t+1}/c_t)}{c_{t+1}/c_t}$$

And

$$d\,ln(U_{t+1}/U_t) = \frac{d(U_{t+1}/U_t)}{U_{t+1}/U_t}$$

Moreover, by Appendix A

$$d\,ln(U_t/U_{t+1}) = \frac{d(U_t/U_{t+1})}{U_t/U_{t+1}} = -d\,ln(U_{t+1}/U_t)$$

Therefore, EIS can be expressed as follows

$$EIS = \left|\frac{dln(c_{t+1}/c_t)}{dln(U_t/U_{t+1})}\right|$$

EIS measures the relative change of consumption from period t to period $t+1$with respect to the relative change of utility following from the change of consumption. Generally, *EIS* depends on the level of consecutive consumptions. However, there are utility functions for each this quantity is constant. Further, we shall consider the recursive utility that has constant elasticity of intertemporal substitution. In 1983, Epstein and Hynes introduced the following aggregator:

$$W(z1;\, z2) = \left((1-\beta)z_1^{\rho} + \beta z_2^{\rho}\right)^{1/\rho} \tag{1}$$

where $\beta \in [0,1)$ is a subjective discount factor and $\rho \epsilon (0;\, 1)$.

Theorem 1. For the recursive utility described by the recurrence equation

$$U(c) = W(c_1;\, U(2c)) = ((1-\beta)c_1^{\rho} + \beta(U(2c)^{\rho}))^{1/\rho}$$

EIS is constant and equals $\frac{1}{1-\rho}$

Proof. Let us consider

$$U(tc) = \left((1-\beta)c_t^{\rho} + \beta\big(U(t+1c)\big)^{\rho}\right)^{1/\rho}$$

We find that

$$U_t := \frac{\partial U(\,tc)}{\partial c_t} = \left((1-\beta)c_t^{\rho} + \beta\big(U(t+1c)\big)^{\rho}\right)^{\frac{1}{\rho}-1}(1-\beta)c_t^{\rho-1}$$

Hence,

$$U_t = \frac{(1-\beta)\big(U(tc)\big)^{1-\rho}}{c_t^{1-\rho}}$$

In an analogous manner we obtain

$$U_{t+1} = \frac{(1-\beta)\big(U({}_{t+1}c)\big)^{1-\rho}}{c_{t+1}^{1-\rho}}$$

We get

$$\left(\frac{c_{t+1}}{c_t}\right)^{1-\rho}\left(\frac{U({}_tc)}{U({}_{t+1}c)}\right)^{1-\rho} = \frac{U_t}{U_{t+1}}$$

and consequently,

$$\frac{c_{t+1}}{c_t} = \left(\frac{U_t}{U_{t+1}}\right)^{\frac{1}{1-\rho}}\frac{U({}_{t+1}c)}{U({}_tc)} \tag{2}$$

Hence,

$$\frac{d(c_{t+1}/c_t)}{d(U_t/U_{t+1})} = \frac{1}{1-\rho}\left(\frac{U_t}{U_{t+1}}\right)^{\frac{1}{1-\rho}-1}\frac{U({}_{t+1}c)}{U({}_tc)} \tag{3}$$

Moreover, from equation (2) it follows that

$$\frac{U({}_{t+1}c)}{U({}_tc)} = \frac{c_{t+1}}{c_t}\left(\frac{U_{t+1}}{U_t}\right)^{\frac{1}{1-\rho}}$$

Plugging this expression in (3) yields

$$\frac{d(c_{t+1}/c_t)}{d(U_t/U_{t+1})} = \frac{1}{1-\rho}\left(\frac{U_t}{U_{t+1}}\right)^{\frac{1}{1-\rho}-1}\left(\frac{U_{t+1}}{U_t}\right)^{\frac{1}{1-\rho}} = \frac{1}{1-\rho}\frac{c_{t+1}}{c_t}\frac{U_{t+1}}{U_t}$$

Therefore, we conclude

$$\frac{\frac{d(c_{t+1}/c_t)}{c_{t+1}/c_t}}{\frac{d(U_t/U_{t+1})}{U_t/U_{t+1}}} = \frac{1}{1-\rho} \Rightarrow \frac{d\ln(c_{t+1}/c_t)}{d\ln(U_t/U_{t+1})} = \frac{1}{1-\rho}$$

2.2 Consumption-Saving Model

We shall consider a model in discrete time, i.e., $t \in T := \mathbb{N}$. In period $t \in T$ a consumer faces a single good $x \geq 0$. The agent decides how much to consume, i.e., he/she takes $c_t \in A(x_t) = [0;\, x_t]$. The remaining part $y_t = x_t - c_t$ is invested for the next period. The satisfaction of consumption c_t is measured by one-period utility $u(c_t) = c_t$ with $\rho \in (0;\, 1]$.

Moreover, a production function that relates output to input is of the following form

$$x_{t+1} = R_{t+1}y_t = R_{t+1}(x_t - c_t) \tag{4}$$

where (R_t) is a sequence of i.i.d. random shocks with non-negative values. Assume that R has the same distribution R_t and its expected value ER is finite.

It is assumed that the agent uses an aggregator introduced by Epstein and Hynes (1983) in order to aggregate a future utility, say y_2, and a current utility y_1; i.e.,

$$W(z_1, z_2) = \left((1-\beta)z_1^{\rho} + \beta z_2^{\rho}\right)^{1/\rho}$$

where $\beta \epsilon\, [0;\, 1)$ is a subjective discount factor. $J(x)$ is the value function, i.e., the optimal utility in the model with the infinite time horizon, if the initial amount of a good is $x \geq 0$. In addition, the decision maker uses the following: *conditional certainty equivalent* due to Kreps and Porteus (1978) to evaluate future utility, i.e., $z_2 =$

$(E_t(J(X_{t+1})^{\gamma})^{1/\gamma}$.

Here, $\gamma > 0$ is a consumer risk sensitivity coefficient and E_t is the expectation operator given the information up to t-stage.

Put $J_0(x) \equiv 0$ for each $x \geq 0$. Let $J_k(x)$ be the optimal utility at the stage k; if the initial state is x. Due to dynamic programming method [5], we evaluate J_k in recursive manner as follows

$$J_{k+1}(x) = \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta(E_k(J_k(y))^{\gamma})^{\rho/\gamma}\right]^{1/\rho}$$

$$= \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta\left(E_k\left(J_k(R_{k+1}(x-c))\right)^{\gamma}\right)^{\rho/\gamma}\right]^{1/\rho}$$

$$= \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta\left(E\left(J_k(R(x-c))\right)^{\gamma}\right)^{\rho/\gamma}\right]^{1/\rho} \qquad (5)$$

where we make use of (4). Hence, the decision maker his/her utility in the consumption- saving model faces the following decision problems:

1. step: $J_1(x) = \max_{c \in A(x)} (1-\beta)^{1/\rho} u(c) = (1-\beta)^{1/\rho} x.$ The function that maximizes the equation is $c_1(x) = x$.

2. step: put $C := E(R^{\rho})^{\rho/\gamma}$,

$$J_2(x) = \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta(E_1(J_1(y))^{\gamma})^{\rho/\gamma}\right]^{1/\rho}$$

$$= \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta\left(E\left((1-\beta)^{1/\rho}R(x-c)\right)^{\gamma}\right)^{\rho/\gamma}\right]^{1/\rho}$$

$$= \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta(1-\beta)(x-c)^{\rho}E(R^{\gamma})^{\rho/\gamma}\right]^{1/\rho}$$

$$= \max_{c \in A(x)} \left[(1-\beta)xa^{\rho} + \beta(1-\beta)(x-xa)^{\rho}E(R^{\gamma})^{\rho/\gamma}\right]^{1/\rho} \quad \text{(assume } c(x) = ax\text{)}$$

$$= x(1-\beta)^{1/\rho} \max_{a \in [0,1]} [a^{\rho} + \beta(1-a)^{\rho}C]^{1/\rho}$$

$$= x(1-\beta)^{1/\rho}\left[1 + (\beta C)^{\frac{1}{1-\rho}}\right]^{\frac{1-\rho}{\rho}}$$

the constant a that maximises the right-hand side is $a_2 = \frac{1}{1+(\beta C)^{\frac{1}{1-\rho}}}$ thus $c_2(x) = \frac{x}{1+(\beta C)^{\frac{1}{1-\rho}}}$ (see Appendix);

3. step:

$$J_3(x) = \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta(E_2(J_2(y))^{\gamma})^{\rho/\gamma}\right]^{1/\rho}$$

$$= \max_{c \in A(x)} \left[(1-\beta)c^{\rho} + \beta\left(E\left\{(1-\beta)^{1/\rho}\left[1 + (\beta C)^{\frac{1}{1-\rho}}\right]^{\frac{1-\rho}{\rho}} R(x-c)\right\}^{\gamma}\right)^{\rho/\gamma}\right]^{1/\rho}$$

$$= \max_{c \in A(x)} \left[(1-\beta)(xa)^{\rho} + \beta(1-\beta)\left[1 + (\beta C)^{\frac{1}{1-\rho}}\right]^{1-\rho} (x-xa)^{\rho}E(R^{\gamma})^{\rho/\gamma}\right]^{1/\rho}$$

$$= x(1-\beta)^{1/\rho} \max_{a\epsilon[0,1]} \left[a^\rho + \beta(1-a)^\rho \left[1+(\beta C)^{\frac{1}{1-\rho}}\right]^{1-\rho} C\right]^{1/\rho}$$

$$= x(1-\beta)^{1/\rho} \left[1+(\beta C)^{\frac{1}{1-\rho}} + (\beta C)^{\frac{2}{1-\rho}}\right]^{\frac{1-\rho}{\rho}}$$

the constant a that maximizes the right-hand side is $a_3 = \frac{1}{1+(\beta C)^{\frac{1}{1-\rho}}+(\beta C)^{\frac{2}{1-\rho}}}$.

Hence, by induction we conclude that n step:

$$J_n(x) = x(1-\beta)^{1/\rho} \left[1+(\beta C)^{\frac{1}{1-\rho}} + (\beta C)^{\frac{2}{1-\rho}} + \cdots + (\beta C)^{\frac{n-1}{1-\rho}}\right]^{\frac{1-\rho}{\rho}}$$

and the optimal policy that maximises the decision problem a_t n-the step is

$$a_n = \frac{1}{1+(\beta C)^{\frac{1}{1-\rho}} + (\beta C)^{\frac{2}{1-\rho}} + \cdots + (\beta C)^{\frac{n-1}{1-\rho}}}$$

In order to show the existence of a solution to the optimality equation in the infinite time horizon we need to impose the following assumption:

(A) $\beta^{1/\rho} E(R^\gamma)^{1/\gamma} < 1$.

Note that the sequence $(J_n(x))_n$ is non-decreasing for every $x \geq 0$, since **(A)** implies that $\beta C < 1$. Therefore, $\lim_{n\to\infty} J_n(x)$ exists and by **(A)**

$$J(x) \coloneqq \lim_{n\to\infty} J_n(x) = x(1-\beta)^{1/\rho} \left[\frac{1}{1-(\beta C)^{\frac{1}{1-\rho}}}\right]^{\frac{1-\rho}{\rho}} \qquad (6)$$

Making use of (5) and the form of t the value function in the finite horizon, we obtain

$$J_{n+1}(x) \geq \left[(1-\beta)c^\rho + \beta\left(E\left\{R(x-c)(1-\beta)^{1/\rho}\left[1+(\beta C)^{\frac{1}{1-\rho}}+\cdots+(\beta C)^{\frac{n-1}{1-\rho}}\right]^{\frac{1-\rho}{\rho}}\right\}^\gamma\right)^{\rho/\gamma}\right]^{1/\rho}$$

for every $c\epsilon A(x)$.

By allowing n tend to infinity in the above display we conclude that

$$J(x) \geq \left[(1-\beta)c^\rho + \beta\left(E\left\{R(x-c)(1-\beta)^{1/\rho}\left[\frac{1}{1-(\beta C)^{\frac{1}{1-\rho}}}\right]^{\frac{1-\rho}{\rho}}\right\}^\gamma\right)^{\rho/\gamma}\right]^{1/\rho}$$

$$= \left[(1-\beta)c^\rho + \beta\left(E\{J(R(x-c))\}^\gamma\right)^{\rho/\gamma}\right]^{1/\rho}$$

for every $c\epsilon A(x)$.: Hence, it holds

$$J(x) \geq \max_{c\epsilon A(x)} \left[(1-\beta)c^{\rho} + \beta\left(E\{J(R(x-c))\}^{\gamma}\right)^{\rho/\gamma}\right]^{1/\rho} \qquad (7)$$

for all $x \geq 0$.

Now consider the problem of maximization in (7). Again, suppose that the function that maximizes right side has the form $c^*(x) = a^*x$: Hence, max

$$\max_{c\epsilon A(x)} \left[(1-\beta)c^{\rho} + \beta\left(E\{J(R(x-c))\}^{\gamma}\right)^{\rho/\gamma}\right]^{1/\rho} =$$

$$\max_{c\epsilon[0,1]} x(1-\beta)^{1/\rho}\left[a^{\rho} + \beta\left(E\left\{R(1-a)\left(\frac{1}{1-(\beta C)^{\frac{1}{1-\rho}}}\right)^{\frac{1-\rho}{\rho}}\right\}^{\gamma}\right)^{\rho/\gamma}\right]^{1/\rho} =$$

$$\max_{c\epsilon[0,1]} x(1-\beta)^{1/\rho}\left[a^{\rho} + \beta(1-a)^{\rho}\left(\frac{1}{1-(\beta C)^{\frac{1}{1-\rho}}}\right)^{\frac{1-\rho}{\rho}}\{ER^{\gamma}\}^{\rho/\gamma}\right]^{1/\rho} =$$

$$\max_{c\epsilon[0,1]} x(1-\beta)^{1/\rho}\left[a^{\rho} + (1-a)^{\rho}\frac{\beta C}{\left(1-(\beta C)^{\frac{1}{1-\rho}}\right)^{1-\rho}}\right]^{1/\rho}$$

From Appendix A, it follows that

$$a^* = \frac{1}{1+\left(\frac{\beta C}{\left(1-(\beta C)^{\frac{1}{1-\rho}}\right)^{1-\rho}}\right)^{\frac{1}{1-\rho}}} = \frac{1}{1+\frac{(\beta C)^{\frac{1}{1-\rho}}}{1-(\beta C)^{\frac{1}{1-\rho}}}} = 1-(\beta C)^{\frac{1}{1-\rho}}$$

Note that $a^* = \lim_{n\to\infty} a_n$.On the other hand, plugging $c^*(x)$ in (7) we obtain that

$$J(x) \geq x(1-\beta)^{1/\rho}\left[a^{*\rho} + (1-a^*)^{\rho}\frac{\beta C}{\left(1-(\beta C)^{\frac{1}{1-\rho}}\right)^{1-\rho}}\right]^{1/\rho}$$

$$= x(1-\beta)^{1/\rho}\left[\left(1-(\beta C)^{\frac{1}{1-\rho}}\right)^{\rho} + (\beta C)^{\frac{\rho}{1-\rho}}\frac{\beta C}{\left(1-(\beta C)^{\frac{1}{1-\rho}}\right)^{1-\rho}}\right]^{1/\rho}$$

$$= x(1-\beta)^{1/\rho}\left[\left(1-(\beta C)^{\frac{1}{1-\rho}}\right)^{\rho} + \frac{(\beta C)^{\frac{1}{1-\rho}}}{\left(1-(\beta C)^{\frac{1}{1-\rho}}\right)^{1-\rho}}\right]^{1/\rho}$$

$$= x(1-\beta)^{1/\rho}\left[\frac{1}{1-(\beta C)^{\frac{1}{1-\rho}}}\right]^{\frac{1-\rho}{\rho}} = J(x)$$

Summarising our considerations, we can formulate the following result.

Theorem 2. Assume (A). Then, the function $J(x)$ in (6) is a solution to the optimality equation, i.e., it holds

$$J(x) = \max_{c\in A(x)}\left[(1-\beta)c^{\rho} + \beta\left(E\left(J\big(R(x-c)\big)\right)^{\gamma}\right)^{\frac{\rho}{\gamma}}\right]^{1/\rho}$$

and the function $c^*(x) = x\left(1-\left(\beta E(R^{\gamma})^{\rho/\gamma}\right)^{\frac{1}{1-\rho}}\right)$ that attains the maximum in the above equation constitutes the optimal policy, i.e., at each step it is optimally to consume

$a^* = 1-\left(\beta E(R^{\gamma})^{\rho/\gamma}\right)^{\frac{1}{1-\rho}}$ of a current stock for investor.

The value function J(x) can be obtained as a limit of the sequence of value functions obtained in the models with finite time horizon.

3. Examples

In this section, we draw a function

$$\rho \rightarrow 1-\left(\beta E(R^{\gamma})^{\rho/\gamma}\right)^{\frac{1}{1-\rho}}$$

for different values of ρ. Let us set $a^*(\rho) = 1-\left(\beta E(R^{\gamma})^{\rho/\gamma}\right)^{\frac{1}{1-\rho}}$. As argued in ([8]) the coefficient $\gamma\epsilon(0,1)$ is responsible for the risk attitude of the agent. If γ is close to one, then the decision maker becomes risk neutral. If, on the other hand, γ is close to zero, then the decision maker becomes risk averse.

We consider two examples.

R has the uniform distribution $U(0{,}5;\ 1{,}6)$. Then, $ER = 1{,}05$ and $VarR = 0{,}1$.

Thus

Fig.	Risk coefficient γ	ρ	β
1	$\gamma = 0{,}5$	$\rho = 0{,}3$ $\rho = 0{,}5$	0,95
2	$\gamma = 1$ $\gamma = 0{,}1$	$\rho = 0{,}5$	0,95

In both cases, the red curve is above the blue one. In other words, the risk averse agent prefers to consume more and invest less when compared with the agent that cares less about the volatility of the future payoff. Moreover, two instances show that all functions $a^*(\rho)$are increasing. Clearly, when ρ increases, then *EIS* does as well. Distance between curves is big. A lower *EIS* value causes the consumption to be twice as high. We can see the significant influence of this factor on the optimal strategy. In the Drawings 2 the impact of the risk factor on consumption strategies was analyzed (investment), when the elasticity coefficient of intertemporal substitution is constant and equals 2. The red curves reflect the case of a risk sensitive investor, and the blue curves are plotted for a risk neutral trader. In all cases, the red curves lie above the blue curves. Value differences the γ coefficient cause smaller differences in the consumption value than the change in value the coefficient ρ.

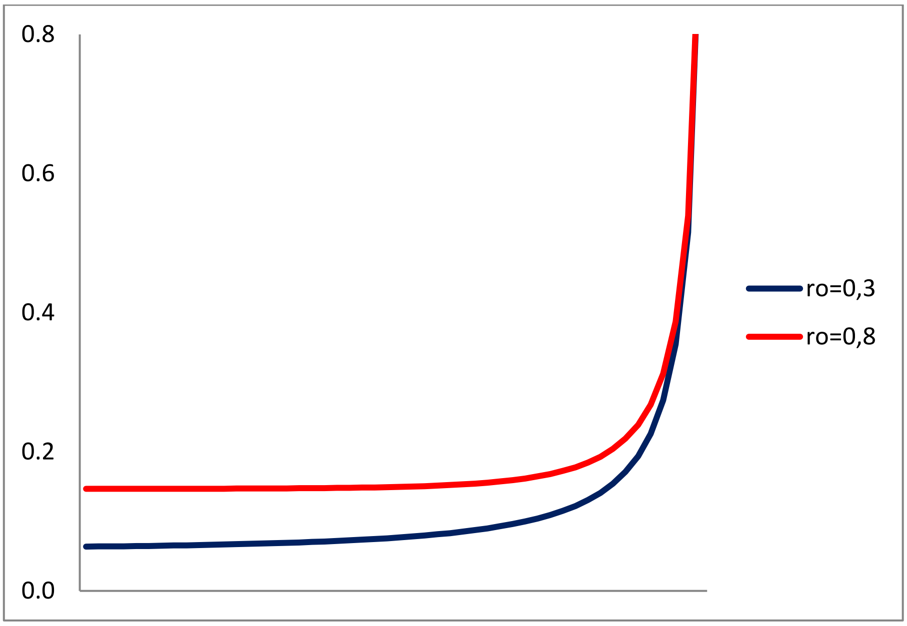

Figure 1 The function $\rho \to a^*(\rho)$

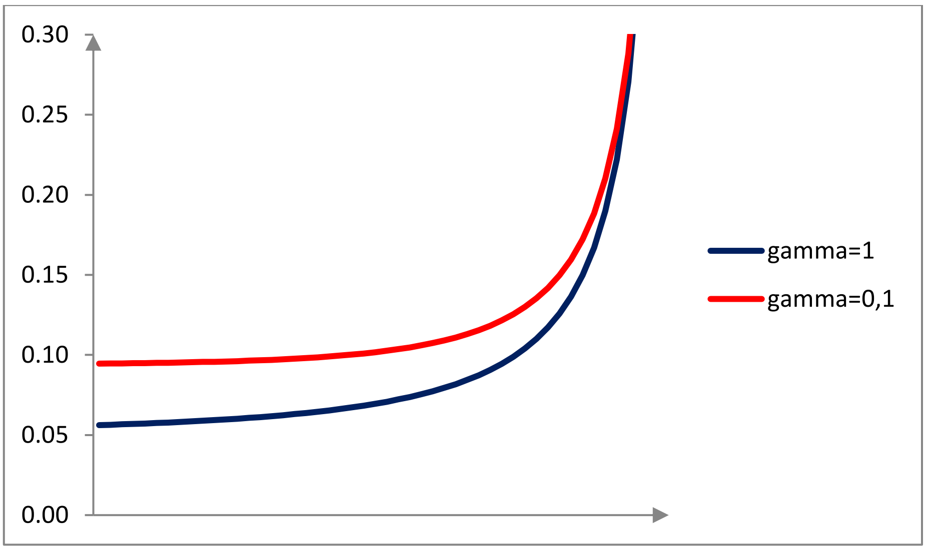

Figure 2 The function $\rho \to a^*(\rho)$

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Appendix

A. Logarithmic Derivative

Let $f: R \rightarrow R$ be a differentiable function. Then, we have

$$\frac{dlnf(x)}{dx} = \frac{f'(x)}{f(x)}$$

Hence,

$$dlnf(x) = \frac{f'(x)dx}{f(x)} = \frac{df(x)}{f(x)} \Longrightarrow dlnf = \frac{df}{f}$$

Moreover,

$$\frac{dln(1/f(x))}{dx} = -\frac{1}{1/f(x)}\left(\frac{1}{f(x)}\right)^2 f'(x) = -\frac{f'(x)dx}{f(x)}$$

and consequently,

$$dln(1/f) = -\frac{df}{f}$$

B. Function Maximization

In our examples we deal with the maximization of the following function:

$f(x) = (ax^{\rho} + b(1-x)^{\rho})^{1/\rho}$ where $\rho \epsilon\ (0;\ 1),\ a, b > 0, x \in [0;\ 1].$

Since the function $z \rightarrow z^{1/\rho}$ is increasing (because $1/\rho > 0$), it suffices to consider the maximisation of the function

$$f_0(x) = ax^{\rho} + b(1-x)^{\rho}$$

The first order condition yields

$$f_0'(x) = \rho a x^{\rho-1} + \rho b(1-x)^{\rho-1} = 0$$

Hence, for $x \neq 1$ and $x \neq 0$ it follows that

$$\frac{a}{x^{1-\rho}} = \frac{b}{(1-x)^{1-\rho}} \Longrightarrow (1-x)a^{\frac{1}{1-\rho}} = xb^{\frac{1}{1-\rho}}$$

Hence,

$$x_{max} = \frac{a^{\frac{1}{1-\rho}}}{a^{\frac{1}{1-\rho}} + b^{\frac{1}{1-\rho}}} = \frac{1}{1-\left(\frac{b}{a}\right)^{\frac{1}{1-\rho}}}$$

The function f_0 is the sum of two strictly concave functions, and therefore, f_0 is strictly concave. Thus, the function f at x_{max} attains the maximum.