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Approximate Solution of Stress Distribution in the Beam loaded on Bending in the Steady-State Creep Conditions

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Abstract: The straight prismatic beam loaded on bending is considered at the paper. The beam is uniformly heated and it is situated in the state of *steady-state creep.* If the bending moment, M_y acts on the beam, the normal stresses, σ_x, appear which will be *constant* over the time. Those stresses are *non-linearly* distributed across the cross-section of the beam, what was investigated and published in the earlier papers. In this paper, the stresses are analyzed by means of one *variation method - the principle of the minimum of complementary potential energy* accumulated in the beam. The solutions, obtained by that method will be approximate but enough accurate, and give the *linear* distribution of the normal stresses, $\sigma_{x,}$ across the cross-section of the beam. They are reliable and good approximation of the exact analytical solution.

Key words: non-linear and linear stress distribution, beam loaded on bending, steady-state creep, variation method, principle of minimum complementary potential energy

1. Introduction

The distribution of the normal stresses, σ_x, across the cross-section of a prismatic beam, loaded on bending and at the *steady-state creep* conditions was determined exactly, analytically, and the obtained results were published in the paper [1]. In these articles, those results are repeated, on the Fig. 1, in a slightly modified form. The results are related to the rectangular cross-section of a beam. It is seen, from the Fig. 1, that the normal stresses, σ_x, are markedly *non-linearly* distributed across the whole cross-section of the beam. The non-linearity is especially stressed at the closeness of the neutral axis, y, i.e., for the small values of the independent variable, z. The material parameter n, which appears in the *Norton's constitutive equation* [2, 3] assumes the values n = 2, 4, 6, 8 and 10. The obtained stress distribution is *equivalent* to the distribution which will be obtained at the *non-linear elastic* material of the beam. It means that in the analysis of stress distribution some of the *variation methods* [4], could be applied. The *principle of minimum complementary potential energy* will be applied in this paper. The results obtained by that approximate variation method will be compared with those obtained by the exact analytical method [1].

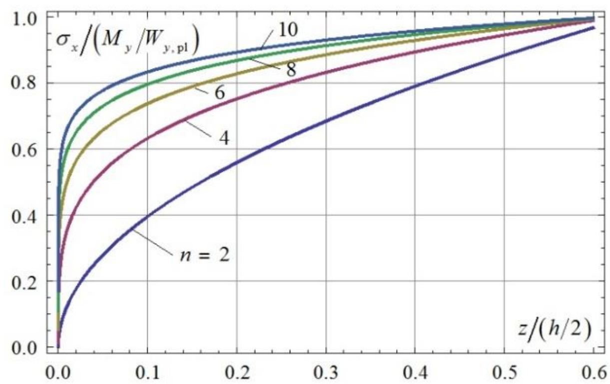

Fig. 1 Distribution of non-dimensional normal stresses across the cross-section of the beam in the stationary creep conditions.

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From the diagram at the Fig. 1, it can be noticed that maximum normal stress, $\sigma_{x.max}$, which appears on the outer boundaries of beam, is *less* in comparison to one which appears in the initial moment of creeping, and that it *decreases* as the parameter *n increases*. Similarly, it can be noticed that the stresses become more uniformly distributed across the whole cross-section of the beam with increase of the parameter *n*.

2. Complementary Potential Energy Accumulated in the Beam

The density of complementary potential energy in the beam loaded on bending amounts to, according to [4]

$$\Pi_0 = \sigma_0^2 \Big/ 2K + \int_0^{\sigma_e} \varepsilon_e \cdot \mathrm{d}\sigma_e . \qquad (1)$$

The first summand represents the *density of hydrostatic strain energy*, while the second one is equal to the area hatched by the horizontal lines, according to the Fig. 2. By bending the beam in the steady-state creep conditions, the uniaxial state of stress appears, for which is valid $\sigma_e = \sigma_x$ and $\varepsilon_e = \varepsilon_{e.c} = \varepsilon_{x.c}$. Also, it is assumed that at the beam creeping there is no the change of its volume (the beam material is *incompressible*) and $K = \infty$. The first summand in the expression (1), in that case, becomes equal to zero. Furthermore, if instead of, $\varepsilon_{e.c}$, the *Norton's law* is inserted [2], according to which is

$$\varepsilon_{e.c} = \sigma_e^n \cdot \Omega(t,T), \qquad (2)$$

then the expression (1) changes to

$$\begin{aligned} \Pi_0 &= \int_0^{\sigma_x} \sigma_x^n \cdot \Omega(t,T)\,\mathrm{d}\sigma_x = \\ &= \left[\sigma_x^{n+1} \big/ (n+1)\right] \cdot \Omega(t,T). \end{aligned} \qquad (3)$$

The *complementary potential energy* accumulated in the whole beam of length, *l*, is determined by integrating across the volume of beam

$$\begin{aligned} \tilde{\Pi} &= \int_V \Pi_0 \cdot \mathrm{d}V = \int_V \frac{\sigma_x^{n+1}}{n+1} \cdot \Omega(t,T) \cdot \mathrm{d}V = \\ &= \frac{2lb}{n+1} \cdot \Omega(t,T) \cdot \int_0^{h/2} \sigma_x^{n+1} \cdot \mathrm{d}z. \end{aligned} \qquad (4)$$

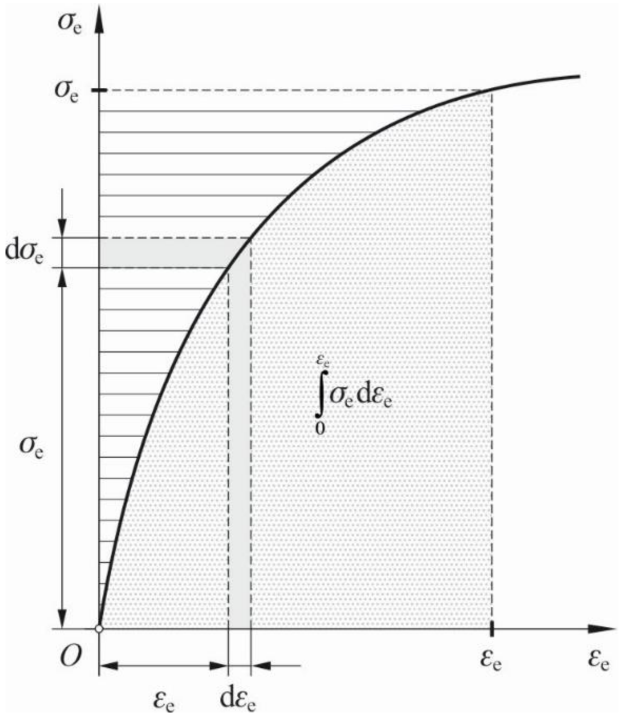


Fig. 2 Equivalent stress-strain diagram of beam material and the density of complementary potential energy.

3. Approximate Solution of Stress Distribution in the Steady-State Creep Conditions

The complementary potential energy, according to the analytical expression (4), is not possible to determine at this moment because the stress distribution, σ_x, across the cross-section of the beam is, by now, unknown. Therefore, that stress distribution is necessary to *assume*. So, according to the expression which is possible to find in the book [4], for the case of *steady-state creeping*, the *approximate solution* looks like

$$\sigma_x = \sigma_{x.1} + k \cdot (\sigma_{x.el} - \sigma_{x.1}). \qquad (5)$$

This analytical solution was, firstly, suggested by the Russian scientist L. M. Kachanov for the case of *steady-state creep*. Namely, in the case of *stationary* *creep* the stresses are *constant over time.* Similarly, the stresses are unchanged *with time* in the problems of theory of plasticity. The above expression (5) represents the superposition of the two states of stresses: one which corresponds to the limit plastic state in which it is assumed that the beam material is *elastic-perfectly plastic*, Fig. 3b, and second in which the *linear* distribution of the normal stresses is assumed, according to the Fig. 3c. In the first case the stresses are calculated according to the formula $\sigma_{x.1} = M_y / W_{y.\mathrm{pl}} = 4M_y / bh^2$, while in the second case the stresses are determined from the expression for the elastic range, i.e., $\sigma_{x.\mathrm{el}} = 12M_y \cdot z / bh^3$.

When those expressions are inserted in the formula (5) it is obtained

$$\sigma_x = \frac{4M_y}{bh^2} \cdot \left[1 + k \cdot \left(3 \cdot \frac{z}{h} - 1\right)\right] \tag{6}$$

The *linear* dependence of the stresses, σ_x, upon the variable, *z*, is easily noticed from the expression (6). By superposing the diagrams at the Figs. 3b and 3c, the resulting diagram of the stress distribution in the beam is achieved in the case of *stationary creep*, according to the Fig. 3f. The solution is approximate and reminds on the stress distribution in the beam in the theory of plasticity in the case of *elastic-linear strain hardening material.* The qualitative illustration of the stress distribution, σ_x, is presented diagrammatically on the Figs. 3b to 3f.

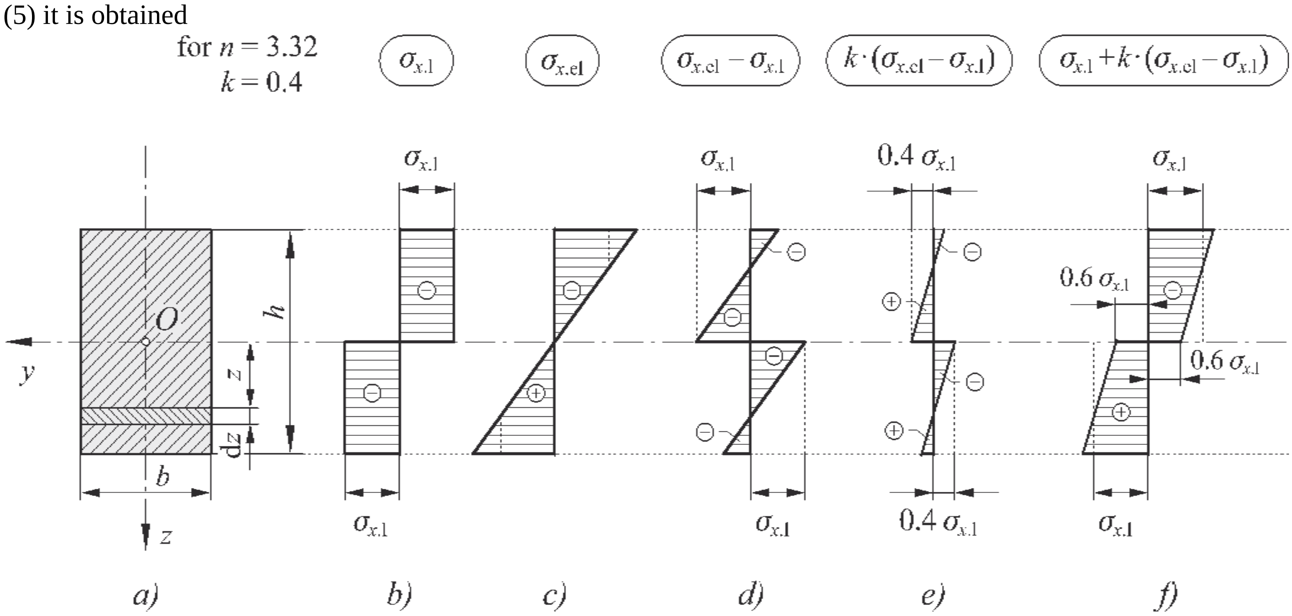


Fig. 3 Distribution of the normal stresses, σ_x, across the cross-section of the beam in the *steady-state creep conditions*, according to the expression (5) as L. M. Kachanov suggested: a) cross-section of the beam, b) case of limit plastic state, c) case of linear elastic state, f) resulting diagram which approximates the stress distribution in the case of steady-state creep.

4. Principle of Minimum Complementary Potential Energy

The expression (6) is inserted in (4) and it is obtained

$$\tilde{\Pi} = \frac{2lb}{n+1} \cdot \Omega(t,T) \cdot \int_0^{h/2} \left(\frac{4M_y}{bh^2}\right)^{n+1} \cdot \left[1 + k \cdot \left(3 \cdot \frac{z}{h} - 1\right)\right]^{n+1} \cdot \mathrm{d}z. \tag{7}$$

Now, the *principle of minimum complementary potential energy* is applied, according to Ref. [4], i.e., it is looked for

$$\partial \tilde{\Pi} / \partial k = 0. \tag{8}$$

According to that principle, of all possible static states of stresses, the complementary potential energy accumulated in the whole beam will have the *minimum* value, only in the case of true state of stress. So, from the expression (7) and the condition (8), after arranging, the *non-linear algebraic equation*, $k = k\,(n)$, is obtained, from which the values of proportionality factor, k, is calculated in dependence of parameter, n. The equation looks like

$$\left[1-(n+1)\cdot k/2\right]/\left[1+(n+1)\cdot k\right]= \\ =\left[(1-k)/(1+k/2)\right]^{n+1}. \tag{9}$$

5. Numerical Solution of the Non-Linear Algebraic Equation

The Eq. (9) is markedly non-linear. It is possible to solve it by some mathematical program package. In this paper the program package *Wolfram Mathematica 7.0* was used [7]. For the discreet values of the material parameter n = 1, 2, 3,…, 14, the values of proportionality factor, k, were determined numerically, respectively, the inverse procedure was applied, i.e., for assumed values of factor, k, the parameter, n, was determined. The results of computation are presented in the Table 1. On the basis of that results the graph of the function, $k = k\,(n)$, was constructed and it is presented at the Fig. 4.

Table 1 For the assumed values of factor, *k*, the value of material parameter, *n*, is determined, numerically.

k [-]	0.1	0.14	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
n [-]	14	10	6.95	4.546	3.32	2.60	2.082	1.717	1.444	1.20	1

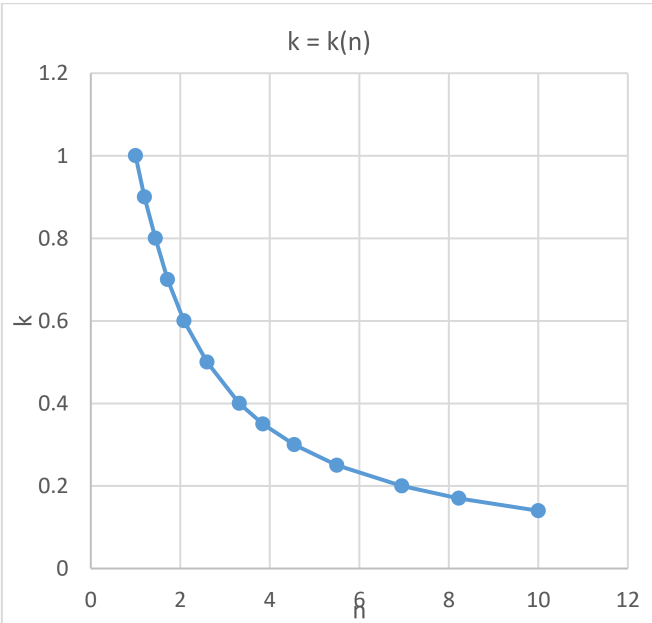


Fig. 4 Diagrammatic presentation of dependence of proportionality factor, *k*, on the material parameter, *n*.

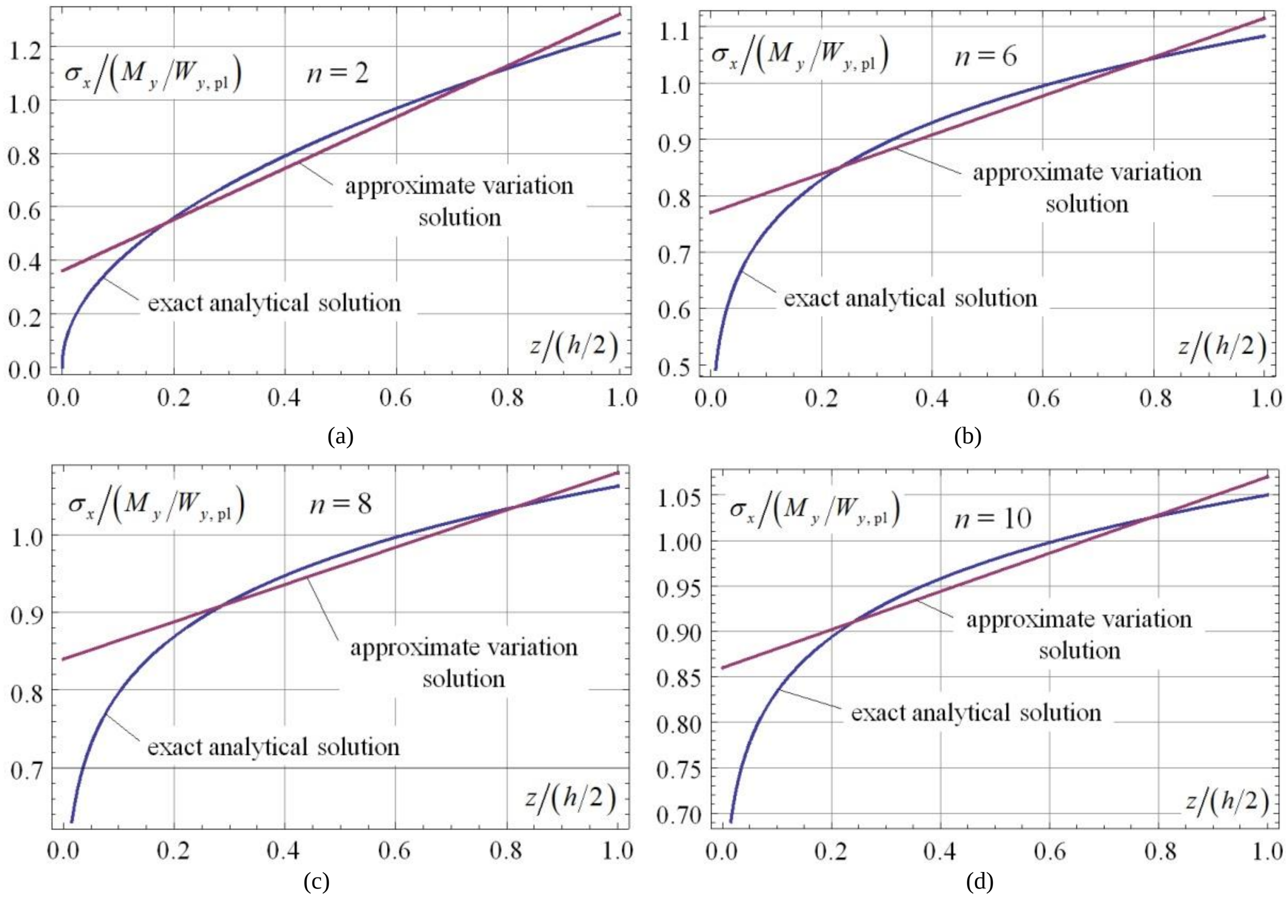
6. Comparison of Exact Analytical and Approximate Variation Solution for Stress Distribution

Normal stress distribution, σ_x, across the cross-section of prismatic beam, loaded on bending and in the *stationary creep conditions,* was determined exactly, analytically, and the results were presented diagrammatically in the papers [1, 9]. The same results were repeated once again in this paper, on the Fig. 1, so they could be compared with the results achieved by approximate analytical method, according to the expressions (5) and (6). Only the final expressions needed for computing and drawing the curves presented at the Fig. 1 are quoted. Those formulae were derived in the papers [1, 9] and here only their final forms are quoted, in the Table 2, according to which the curves, at the Fig. 1, were calculated and drawn, using the program package *Wolfram Mathematica 7.0*, [7].

The normal stress distribution, σ_x, across the cross-section of the beam, according to the variation method, is calculated according to the approximate expression (6). That method gives the *linear* dependence of the stresses, σ_x, on the variable, z. The values of the proportionality factor, k, are taken from the Table 1, or they are red off from the Fig. 4 for the discreet values of the material parameter n = 2, 4, 6, 8 and 10. Finally, the comparison of the exact analytical and approximate variation solution which is, at the same time, *linear*, is presented at the Figs. 5a to 5d. It is seen that this approximate solution excellently approximates the exact, *non-linear*, analytical solution.

Table 2 Analytical expressions for an exact computing and drawing the diagram of stress distribution, derived in the paper [1].

$$\sigma_x = M_y \cdot z^{1/n} / I_{ny}; \qquad I_{ny} = \alpha_1 \cdot b \cdot h^{(2n+1)/n}; \qquad \alpha_1 = \left(1/2^{(n+1)/n}\right) \cdot \left(n/(2n+1)\right).$$



(a) (b) (c) (d)

Fig. 5 Comparison of the exact analytical and the approximate variation solution for the stress distribution in the beam in the stationary creep conditions.

7. Conclusion

The prismatic beam loaded on bending was considered in this paper, which is situated in the state of *stationary creep*. The stress distribution in a such beam and at the such conditions is possible to determine exactly, analytically, what was performed and indicated in the paper [1] and in some modified form is again presented at this paper, on the Fig. 1. As it can be seen from the Figure, that distribution is markedly *non-linear* with great gradients, especially in the middle of cross-section of the beam, around the neutral axis, *y*.

Therefore, the prime task of this paper was to determine the stress distribution by some approximate, *variation method* which will be enough accurate and reliable and which will give, eventually, the *linear* stress distribution for the steady-state creep conditions of the beam. Therefore, the approximate solution was taken in the form as it was suggested by Russian scientist L. M. Kachanov in the book [4]. It is presented by the analytical expression (5), and the qualitative stress distribution, according to that solution, is presented diagrammatically on the Fig. 3f. In order to achieve that solution a *variation method* was applied which is founded on the *principle of minimum complementary potential energy* [4]. The comparison of the exact analytical and the approximate variation solution for the stress distribution in the beam is presented at the Figs. 5a to 5d, for the discreet values of the material parameter n = 2, 6, 8 and 10. It is seen that the approximate solution excellently approximates the exact analytical solution, except in the immediate closeness of the neutral axis, *y*. Namely, at that place, at the approximate solution, the discontinuance in the stress distribution exists, what is seen from the diagram at the Fig. 3f. To conclude, the variation method gives the solution which is enough accurate and reliable for the engineering calculations, and as it is *linear*, it means that the performance of the analysis, i.e., the numerical procedure, is simple.

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