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The Church-Turing-Chaitin Thesis in the Behavior of Dynamic Economic Phenomena

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Abstract: Accepting the classical-deterministic-conception implies assuming as an starting point an axiomatic according to which every dynamic economic system can be formalized in the assumption that, in the limit, its behavior obeys a trajectory that is the necessary consequence of the influence of infinite factors that constitute its causal structure and whose mathematical modeling, inevitably incomplete, generates a type of uncertainty comparable to a randomness synonymous with ignorance. On the contrary, the thermodynamic-aleatorist- conception implies replacing the axiom of existence of a trajectory by the assumption of random behavior that may eventually generate certain local regularities at the macroscopic level and, consequently, substitute the classical interpretation of randomness due to ignorance by that of absolute-randomness.

The objective of this paper is to contribute to the solution of the historical question of the fundamentals of the nature of the randomness inherent in the behavior of dynamic economic phenomena trough the formal demonstration of the truth or falsity of the deterministic or aleatorist conceptions using the arguments which provides algorithmic complexity theory and the assimilation of its modeling to the formal field of mathematics.

Key words: dynamic phenomena, aleatorism, determinism, Church-Turing-Chatin thesis, algorithmic complexity

JEL codes: B41, B23, C18

1. An Introduction to the Interpretation of the Behavior of Dynamic Phenomena

Let a dynamic phenomenon $Y(t,w)$ be assimilable to a stochastic process that evolves in the time domain $(t \in T)$ and whose configuration varies in the domain of the states (phases or variables), $w \in \Omega(Y)$, in which it is assumed that each state $\mathrm{w(t)}$ is defined by the simultaneous realization, at time t, of the infinite random variables that form its causal structure:

$$\Omega\big(Y(t)\big) = \{Y(t-j), X_1(t-h_1), X_2(t-h_2), X_3(t-h_3), \dots\}$$

(for $j, h_1, h_2, \dots \in \mathbb{R}, j > 0, h_i \geq 0, i = 1,2, \dots$) and the time succession of the states forms their necessary trajectory. Then, the phenomenon Y(t,w) is defined as an entity that evolves in a space-time ambit characterized

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by an immutable chain of causes and consequences.

The deterministic interpretation of the behavior of *Y(t,w)* (at least at the macroscopic level) is based on the assumption that the process constitutes an axiomatic system, originated in certain premises of a metaphysical order: i) that the field to which the phenomena belong it is real; ii) that there are objective laws that govern their behavior and iii) that these laws are inherent to phenomena, rational and asymptotically knowable[1].

This assumption of universal solidarity that causally relates to phenomena -formally represented by the consideration of $\Omega\big(Y(t)\big)$ as an infinite numerable set- and makes the nature of $\{Y(t)\}$ appear infinitely complicated, allows concluding that its representation from a finite subset of causal variables, $\Omega^*\big(Y(t)\big) \subset \Omega\big(Y(t)\big)$, defined by a function (non-stochastic) $f\big[\Omega^*\big(Y(t)\big)\big]$, will inevitably be insufficient, that is, part of its behavior will remain unexplained. So, under certain conditions of stationarity it can be written:

$$Y(t) = f\big[\Omega^*\big(Y(t)\big)\big] + \varepsilon(t)$$

where: i) $f\big[\Omega^*\big(Y(t)\big)\big]$ denotes the formal behavior that Y(t) would be expected to observe if the factors included in $\Omega^*\big(Y(t)\big)$ were its only causes, and f[·] was the function that represented the true time invariant causal relationship between these factors and $Y(t)$; ii)

$$\lim_{\Omega^*(Y(t))\to\Omega(Y(t))} f\big[\Omega^*\big(Y(t)\big)\big] = Y(t)$$

and iii) $\varepsilon(t)$ denotes the randomness-ignorance component generated by the difference between $\Omega^*\big(Y(t)\big)$ and the causal structure $\Omega\big(Y(t)\big)$:

$$\varepsilon(t) = Y(t) - f\big[\Omega^*\big(Y(t)\big)\big]$$

and, consequently, is such that:

$$\lim_{\Omega^*(Y(t))\to\Omega(Y(t))} \varepsilon(t) = 0$$

This notion of randomness-ignorance can be considered masterfully synthesized in the following text due to Poincaré (1905): "*Every phenomenon, no matter how small, has a cause and an infinitely powerful spirit wonderfully well informed of the laws of Nature, it could have to foresee from the beginning of the centuries (…) For him the word randomness would have no meaning or, rather, there would be no randomness. It is because of our weakness and our ignorance that randomness exists for us (…) Randomness is nothing more than the measure of our ignorance. Fortuitous phenomena are, by definition, those of which we are ignorant of its laws*" (p. 204).

In this paradigm, the trajectory of $Y(t)$ would admit a representation in terms of reversible classical mechanics in the time domain, in which time would elapse uniformly and, consequently, the difference between the past and the future would have no meaning, in which knowing the present state of the phenomenon its past could be reproduced and its future calculated in a deterministic way; that is to say, a representation in which the present would be but a point that separates "*æternitas a parte ante ed æternitas a parte post*".

This Thomistic notion of randomness-ignorance, which dominated the panorama of the philosophy of science until the formulation of quantum mechanics at the beginning of the 20th century, had a great boom in the 18th century with the consecration of Newton's laws of classical mechanics and of Maxwell's electrodynamics. An ideal of perfection, known in the Leibniz nomenclature (1678) as the "*principle of sufficient reason*", which states the "... *equivalence between the 'full' cause and the 'total' effect*". So, if these two elements could be defined, knowledge about the behavior of *Y(t)* would be equivalent to God's knowledge of the world. A god like that of

[1] No foundation can be attributed to this axiomatic (neither inductive, deductive, nor abductive). According to Daston (1988), in the interpretation of Hume (1718), it constitutes "... *a psychological necessity, an almost involuntary precept implanted by beneficent nature to compensate for the shortcomings of human reason*" (p. 202).

Poincaré, a god who governs a nature " ... *in which there is no place for the non-formalizable*" according to Thom, a god who does not play dice according to Einstein, a god who simultaneously knows the position and speed of a particle, according to Planck, or a demon capable of calculating the past and future of the universe from one's knowledge any of its instantaneous states according to Laplace, a demon capable of reversing the irreversible evolution associated with the growth of entropy from the action on each particular molecule according to Maxwell.

The insufficiency of this representation in terms of classical mechanics to explain "... *an unstable world that we know through a finite window*" (Prigogine & Nicolis, 1977, p. 16), in which the natural state of open systems is a type of non-equilibrium state -a constructive non-equilibrium that, as a consequence of its fundamental property of self-organization, generates new states and new complex structures that are only imaginable in the field of temporal irreversibility- gave rise to a new thermodynamic formulation -aleatorist- whose difference with classical dynamics lied essentially in: i) the postulation of the concept of the state of the process at a given instant as a result of a time-oriented evolution in which, unlike the past and the present, the future it is formed by a succession of causally linked non-observable random variables; ii) the conception of $f\left[\Omega^{*}\left(Y(t)\right)\right]$ as the representation of certain apparent local regularities observed and iii) the substitution of the classical interpretation of $\varepsilon(t)$ as randomness-ignorance (epistemological) by its interpretation as absolute-randomness (ontological), generated by the innovations to which the factors included in the $\Omega\left(Y(t)\right)$ system are subject.

The thermodynamic explanation attempted to define nature in terms of becoming, giving an intrinsic meaning to the "*arrow of time*"[2], substituting the interpretation of "*dynamic time*" as "*time of the fall of the bass*" in "*thermodynamic time*"[3] and incorporating, consequently, two elements: time and history, in a universe that Newton and the classical physicists had assumed to be eternal[4].

Then, given an open system, $Y(t)$, the thermodynamic explanation of its behavior is assimilable to the variation of its entropy, which can be disaggregated into two components: **i**) the entropy exchange between the system, defined by the structure of $Y(t)$ and the outer universe that contains it and **ii**) the influence exerted by the irreversible processes that occur within the system.

This interpretation allows postulating, in a secondary way, a thermodynamic origin of the indeterminism by novelty –introduced by Gustav Theodor Fechner, in which there is a kind of coexistence between an inherent randomness of the behavior of individual systems and a global determinism, between a randomness generated by the innovations and a determinism defined by the mechanisms used by the system to incorporate these innovations (Fechner, 1851, 1866, 1871, 1897).

2. Complexity and the Hilbert-Ackermann's Thesis

The first ideas about complexity — and, consequently, about randomness understood as maximum complexity — were proposed by Leibniz in his "*Discours de Métaphysique*" (1686). In this paper, in what could be considered a questioning of the aforementioned deterministic principle of sufficient reason, Leibniz proposed the distinction, in the field of the sciences of ideas, between systems that admit an axiomatically formal expression — which can be explained by a law that allows describing relationship structures between the variables

[2] According to Eddington's (1928) expression.
[3] According to Serres's (1975) expression.
[4] Newton (1687): "*Absolute, true and mathematical time, by itself and by its very nature, flows freely without any relation to the outside and it is called 'Duration'*" (p. 10).

involved and constructing functional representations of their behavior- and *"irregular"* systems[5].

As an approximation to Leibniz's proposition, in 1928 David Hilbert and Wilhelm Ackermann (Hilbert, Ackermann, 1928; Hilbert, 1929), based on a formalist position, proposed — in the context of what Chaitin (2005) called "*metamaths*"[6] — the *"Entscheidungsproblem*" that consists of deciding about the possibility of defining a formal, complete and consistent axiomatic system that allows determining the truth or falsity of any proposition in the field of mathematics. That is, the possibility of deciding if a system is *"anticipatory"* (according to the Rosen's nomenclature (1985) and, therefore, if it contains a model in itself.

A formal system is said to be complete if every proposition can be translated into a theorem and vice versa if every theorem represents the translation of a proposition. Likewise, a system is said to be consistent if it does not present contradictory affirmations that can be demonstrated (that is both an affirmation and its negation can be translated into theorems).

Hilbert was convinced that the answer to the *"Entscheidungsproblem"* was positive, that traditional mathematics was logically consistent, that it included two kinds of "truth": one syntactic, inherent exclusively to form and another semantics, inherent to external referents, and that the solution to all its paradoxes and inconsistencies lay in incorporating a semantic meaning into the purely syntactic formal structure in which the truth of the propositions could be demonstrated[7].

This structure is composed up of an axiomatic defined by a finite set of finite sequences of abstract symbols and by a finite set of transformation rules that allow one symbol sequence to be converted into a different one: given an axiom, the process consists of applying a finite succession of transformations that convert it into a finite succession of new sequences, in which each sequence is an axiom or is a derivative of the previous axioms. The ending succession of this sequence constitutes a theorem of the system. If the set of axioms is the same "measure" as the set of theorems, then the application of the succession of transformations is useless. On the other hand, if the set of axioms is smaller than the set of theorems, then the system has algorithmic compression capacity, that is, capacity to admit structural behavior, in other words, to enunciate a theory. Then, if the set of axioms that generates a set of theorems is of a minimum measure (that is, if it defines its optimal syntax), then this measure expresses the measure of the complexity of the formal system.

3. Gödel and the Incompleteness Theorem

In 1931 Kurt Gödel published his incompleteness theorem[8] in which, refuting Hilbert and Ackermann's claim to find a "*theory of everything*" in mathematics and seriously contradicting formalist postulates, he demonstrated that arithmetic is not fully formalizable. Later Gödel (1944) and Rossler (1936) concluded that there is no deductive system whose set of axioms is recursive, that contains the theorems of arithmetic and that is also consistent and complete (Nagel, Newman, 1958; Hofstadter, 1979, Shanker, 1988; Rosen, in Casti, Karlqvist, 1991).

The starting point of the development that led Gödel to this formal logical version of his theorem was the

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5 See Rosen (1985), Bar-Hillel; Wagenaar (1991).

6 "*An introspective field of mathematics in which it is studied what mathematics can and cannot achieve*" (Chaitin, 2005, p. 164).

7 This criterion constitutes the essence of formalist philosophy.

8 In his work "*Die Vollständigkeit des Logischen Funktionenkalküls*" (1930), Gödel presented a succession of theorems, of which the most important are the sixth and the eleventh, which are usually referred to as the first and second incompleteness theorems. See Menger (1994), Berto (2007, 2008).

apparent paradox of Epimenides from Crete which, in this context, can be expressed as *"this assertion is false"*[9]. Gödel's intention was to find a way to express affirmations of this type (self-referential paradoxes) in the field of arithmetic. Now, since Epimenides' assertion involves the notion of truth which, according to Tarski's (1936) demonstration, cannot be captured in the field of a formal system, the only possible solution for Gödel was to replace the (non-formalizable) notion of truth by the (formalizable) notion of *"demonstrability"* and transform the paradox into the proposition *"this assertion is not demonstrable"*. It should be noted that if the assertion is demonstrable, then it is true and therefore not demonstrable, so the assertion and its negation are both demonstrable, implying its inconsistency. On the other hand, if the assertion is not demonstrable, then it is true, that is, it is true but not demonstrable, which implies that the formal system is incomplete.

Generalizing this reasoning, Gödel concluded that in every formal system powerful enough to contain all assertions about natural numbers, there is an assertion that is non-demonstrable from the rules of the system. In other words, it concluded that the answer to the *"Entscheidungsproblem*" is negative.

He further designed the necessary rules to construct a meta-mathematical assertion translatable into the proposition *"arithmetic is consistent"* and demonstrated (regardless of Tarski) that this assertion is not demonstrable, and therefore that the consistency of arithmetic cannot be established exclusively by arguments inherent in the formal system of arithmetic itself (Fefferman, 1986).

4. The Computability Theory

Gödel's results and the formalist assimilation of mathematics to a type of morphogenesis and its consequent identification with a recursively axiomatizable mechanism, that is, with a mechanism capable of generating new behaviors (theorems) from given behaviors (axioms) of according to defined rules, they gave rise to the theory of computability or *"Church-Turing thesis"* according to Kleene's (1952) denomination.

Church's analysis (1934, 1936, 1940) was based on the definition of a system *"effectively computable"* as a synonym of function and on the construction of a grammar linked to a set of transformation rules designed to formalize this definition.

From this construction Church raised his thesis, according to which the class of effectively computable systems coincides with the class of recursive systems. Based on this postulate and the relationship between a formal system and its representations, he also concluded that the answer to the *"Entscheidungsproblem"* was negative, that is, that mathematics was not an anticipatory system[10].

Turing's approach (1936, 1938) to the treatment of Hilbert's problem consisted in the use of the abstract concept of a theoretical machine (purely syntactic) — called "*universal Turing machine*" (UTM) — that allowed characterizing, in a mathematical way, to effectively computable systems. The behavior of the machine was controlled by a program composed of a finite number of instructions[11]. In other words, just as Gödel replaced the intuitive notion of truth with the formalizable concept of demonstrability, Turing replaced the intuitive notion of effective computability with the concept of algorithm. This implies that Turing machines and algorithmic systems have the same computational capacity.

In terms of computability theory, an algorithm can be defined as a program applicable in a Turing machine

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9 This expression is also known as "*Eubulides of Miletus paradox*" or "*liar's paradox*".
10 Kleene (1952) arrived at the same conclusions from the concept of recursive function, see Soare (1996), Bennet (1998).
11 A machine understood as a mathematical idea, not associated with a "hardware".

and the "*Entscheidungsproblem*", as the possibility of finding a syntax, that is, of finding a mechanism that allows deciding, in a finite number of steps, if an algorithmic system is computable (or complete)[12].

Be a Turing machine in its initial state. According to the corresponding set of instructions, it may happen that it reaches its final state and stops producing an "output" $UTM(x)$ or that it never stops (known as the *"halting problem"*). A system $f(\cdot): N \rightarrow N$ is said to be computable in the Turing sense if there is a finite $n = g(\omega)$ in the domain of f (where ω defines an *"input"* at $\sum = \{0,1\}$) for which the Turing machine stops producing an *"output"* $UTM(\omega)$ such that $f(n) = g(UTM(\omega))$[13].

As a corollary of the demonstrations by Kleene (1938) and Church that the set of algorithmic systems is a finite subset of the infinite set of all systems $f(\cdot): N \rightarrow N$ and the rigorous demonstration that the necessary and sufficient condition for a system to be algorithmic is that it be recursive, like Gödel, Turing concluded in the impossibility of solving the UTM halting problem, that is, the impossibility of demonstrating recursion or, what is the same, to demonstrate the computability or completeness of an algorithmic system (result known as the *"Turing thesis"*). He also showed that the concepts of system that admits a formal expression and computable system were equivalent and, consequently, that his thesis and that of Church were equivalent.

5. Computability and Randomness

From Gödel, Kleene and Turing's demonstrations that computable systems — that is systems that posse compression capacity — allow enunciating a theory (understood as a structural behavior that allows inferring its future behavior), Chaitin (1988, 1999, 2001, 2005) proposed to use as measure of the complexity of a system the measure of the smallest program capable of defining its structural behavior (that is, the measure of its optimal syntax[14]), which allows the assimilation of the concepts of algorithmic randomness and incompressibility and the demonstration of the impossibility of determining the existence or not of an optimal syntax (known as the "Chaitin thesis")[15].

The demonstration of the falsity of the Church-Turing-Chaitin thesis would imply, then, the acceptance of the reductionist interpretation, that is the recognition that every dynamic system is complete and that its definition consists exclusively in the determination of its optimal syntax, and the demonstration of its veracity would imply the rejection of this interpretation and the consequent acceptance of the existence of dynamic systems for which *"halting"* is not verified in the UTM or, in other words, the existence of systems for which there is no optimal syntax.

6. Dynamic Economic Representations

Taking into account that, as mentioned in sections 1 and 2, the economic phenomena $Y(t, w)$ — due to the assumption of universal solidarity that relates the factors of their environment ($\Omega(Y(t))$) — depend on causal structures defined by infinitely countable sets, so their behavior is infinitely complex and that the economic models consider the variables involved as variables of the mathematical analysis and the functions that represent

[12] The mathematical concept of algorithm as a finite set of ordered rules that can be executed in a certain number of steps, only emerged in the 1930 as a culmination of research on the foundations of mathematical logic, see Knuth (1985), Davies (2000).
[13] In terms of computational theory the "halting problem" is equivalent to the Hilbert problem.
[14] "*Elegant syntax*" according to the Chaitin (2005) nomenclature.
[15] In terms of Chaitin's thesis, Gödel's theorem can be considered as equivalent to the indemonstrability of the incompressibility of a system.

their causal relationships as functions of the mathematical analysis, it can be concluded that they can be analyzed with arguments belonging to the mathematical formalism.

Therefore, it is possible to propose a treatment of the historical determinism-aleatorism dilemma regarding the interpretation of the behavior of economic phenomena based on the application of metamaths associated with the Church-Turing computability condition and Chaitin's proposition about the definition of an optimal syntax.

This assimilation of the morphogenesis contained in the evolution of a dynamic economic system to the formalist field of mathematics and the proven indemonstrability of computational halting and the determination of the existence of an optimal syntax, allowed concluding in the indemonstrability of both the aleatorist proposition like that of its opposite, the deterministic proposition.

7. Summary and Concluding Remarks

Accepting the classical -deterministic- conception implies assuming as a starting point an axiomatic according to which every dynamic system is formalizable on the assumption that, in the limit, its behavior obeys a trajectory that is the necessary consequence of the influence of infinite factors that constitute its causal structure and whose definition, inevitably incomplete, generates a type of uncertainty comparable to a randomness synonymous with ignorance. On the contrary, the thermodynamic — aleatorist — conception implies replacing the axiom of existence of a trajectory by the assumption of random behavior that can eventually generate certain local regularities at the macroscopic level and, consequently, replacing the classical interpretation of randomness-ignorance with that of absolute-randomness.

The objective of this paper is to contribute to the solution of the historical question of the fundamentals of the nature of randomness inherent in the behavior of dynamic phenomena trough the formal demonstration of the truth or falsity of both deterministic and aleatorist conceptions using the arguments provided by the theory of algorithmic complexity.

Starting from the idea of randomness as maximum complexity — proposed by Leibniz, Hilbert and Ackermann from a formalist point of view raised the problem of the possibility of defining a complete and consistent formal system that would allow determining the truth or falsity of a proposition; that is, the possibility of defining a system such that, given a strictly enunciated problem, it is possible to find an algorithm capable of solving it ("*Entscheidungsproblem*").

Hilbert-Ackerman fundamental hypothesis was to suppose that traditional mathematics was logically consistent, formally assimilable to a type of morphogenesis and, consequently, identifiable with a structure that generates new states (theorems) from given states (axioms) according to defined syntactic rules.

Gödel's results, contrary to this formalist proposal, gave rise to the theory of computability mainly from the works of Kleene, Church, Turing and Post.

The Church-Turing analysis, based on the definition of "*computable*" function, concluded in the negative response to the "*Entscheidungsproblem*", that is, in the impossibility of solving the problem of computational "halting", in other words, finding a general recursively axiomatizable mechanism allows deciding, in a finite number of steps, whether a logical proposition is or not a theorem.

Based on this Church-Turing thesis, Chaitin introduced an interpretation of randomness based on the concept of incompressibility associated with the optimal syntax condition and concluded on the impossibility of determining such condition.

This deductive path defined by the adoption of the arguments of the Church-Turing-Chatin theses, according to which the proposition about the existence of an optimal syntax is indemonstrable, and given the infinitely numerable nature of the causal structure of a dynamic economic system that justifies its assimilation to the formalist field of mathematics, allowed demonstrating the indemonstrability of both the thesis that postulates absolute randomness and that of its opposite, which postulates randomness-ignorance and, consequently, in the impossibility of formally solving the historical dilemma determinism versus aleatorism.

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