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The Differences of Ricci Flow between Perelman and Hamilton about the Poincaré Conjecture and the Geometrization Conjecture of Thurston for Mathematics Lectures

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Abstract: In this paper, it will be given the ideas of the proof the Poincaré conjecture and the geometrization conjecture of Thurston by Perelman who is a Russian mathematician posted the first of a series of three e-prints. Perelman's proof uses a modified version of a Ricci flow program developed by Richard Streit Hamilton. Someone said that the major contributors of this work are Hamilton and Perelman.

Actually Ricci flow was pioneered by Hamilton. But the version of surgery needed for Perelman's argument which is extremely delicate as one need to ensure that all the properties of Ricci flow used in the argument also hold for Ricci flow with surgery. Running the surgery in reverse, this establishes the full geometry conjecture of Thurston, and in particular the Poincaré conjecture as a special and simpler case. This is the excellent thing for the mathematics lecturer to find in his/her mathematics education.

Keywords: Ricci Flow, Poincaré conjecture, Geometrization conjecture, Perelman, Hamilton

1. Introduction

The Poincaré conjecture was one of the most important open questions in topology before being proved. In 2000, it was named one of the seven Millennium Prize Problems which were conceived to record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium. The frontier is still open and most difficult problems, and to recognize achievement in mathematics of historical magnitude. The Poincaré conjecture asserts that any closed three-dimensional manifold such that any loop can be contracted to a point is topologically a three-sphere. This asserts that any closed three-dimensional manifold such that any loop can be contracted to a point is topologically a three-sphere. There came three developments that would play crucial roles in Perelman's solution of the conjecture. But the case of three -manifolds turned out to be the Hardest. This is because in topologically manipulating a three-manifold there are too few dimensions to move problem regions out of the way without interfering with something else. It seemed to be the most difficult, as the continuing series of failed efforts, both to prove and to disprove it, showed. In the meantime, at last this equation can be generalized to three-dimensional case being solved by Grigori Yakovlevich Perelman in 2003. The proof of the Poincare conjecture by Perelman started to talk the definition of the Ricci Flow which was introduced in 1982

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by Richard Hamilton at the MIT lecture theater on a Monday afternoon in April.

2. Poincaré Conjecture and Topology

Even if we call topological several results today were already previously known, it is with Poincaré who was the last universalist, that topology gets its modern form, in particular, regarding the properties of surfaces or higher dimensional spaces. Poincaré introduced the fundamental concept of simple connectedness. The examples of closed surfaces which are compact, without boundary and orientable as follow Figure 1. And there is the theorem "Every closed surface is the boundary of a bretzel". The definition of simple connection is "A surface is called simply connected if every curve on it can be continuously deformed to a single point". The example is the Figure 2. And the torus is not simply connected as Figure 3.

Figure 1 Closed Surfaces Which Are Compact, Without Boundary and Orientable

Figure 2 Definition of Simple Connectedness

Figure 3 The Torus Which Is Not Simply Connected

There is a theorem "Any closed, simply connected surface is homeomorphic to the sphere". Poincaré asked himself whether this theorem was true in dimension higher than two, in particular, in dimension three. A three-dimensional space is locally like our ordinary space topologically, even if we could experience strange situations moving into it for instance, one could exit from a door in a room and find himself entering in the same room, in a way, like walking on a circle, or on the surface of a torus one returns to the starting point. Its structure can actually be even quite more complicated. There exist the three-sphere, the three-torus and several other examples analogous to the surfaces that we have seen. In order to have a classification theorem like the one for surfaces, one of the first conjectures of Poincaré was proposed in 1904, which was "Any closed and simply connected three-dimensional space is homeomorphic to the three-sphere". This question can be generalized to higher dimensions. The case $n \geq 5$ was proved by Smale. The case $n = 4$ was proved by Freedman. The case $n = 3$ was proved by Perelman. The topology of 2-dimensional manifolds or surfaces was well understood in the 19th century. For definitions and other background material, see for example Massey, or Munkres 1975, as well as Thurston 1997. Names in small caps refer to the list of references at the end. There is a simple list of all possible smooth compact orientable surfaces. Any such surface has a well-defined genus $g \geq 0$, which can be described intuitively as the number of holes; two such surfaces can be put into a smooth one-to-one correspondence with each other if and only if they have the same genus. The corresponding question in higher dimensions is much more difficult. Perhaps, Henri Poincare was the first to try to make a similar study of 3-dimensional manifolds. The most basic example of such a manifold is the 3-dimensional unit sphere, that is, the locus of all points (x, y, z, ω) in 4-dimensional Euclidean space which have distance exactly from the origin: $x^2 + y^2 + z^2 + w^2 = 1$. He noted that a distinguishing feature of the 2-dimensional sphere is that every simple closed curve in the sphere can be deformed continuously to a point without leaving the sphere. In 1904, he asked a corresponding question in dimension three. In more modern language, it can be phrased as follows: If a smooth compact 3-dimensional manifold M3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M^3 is homeomorphic to the sphere S^3. He commented, with considerable foresight, “Mais cette question nous entîranerait trop loin”. Since then, the hypothesis that every simply connected closed 3-manifold is homeomorphic to the 3-sphere has been known as the Poincare Conjecture. It has inspired topologists ever since, and attempts to prove it have led to many advances in our understanding of the topology of manifolds.

3. Geometric Flows

A possible line of proof of the conjecture is transforming in some way deformation, surgery-cut and paste, keeping unaltered the topological properties, a hypothetical space which is a counter example to the conjecture, in a three-sphere. All the tries in this direction by means of topological arguments have failed for around a century. The winning approach, still in this setting, happened to be deforming the space by means of evolution laws given by partial differential equations. The great advantage, with respect to the only-topological deformations, is that, by means of methods of analysis, these deformations are quantitative. These kinds of evolutions of geometric objects are called geometric flows in general.

Given $\gamma \subset R^2$ a closed simple curve in the plane, we want that at every time, every point moves with normal velocity equal to the curvature at such point. Analytically, Given a closed simple curve $\gamma = \gamma_0$:

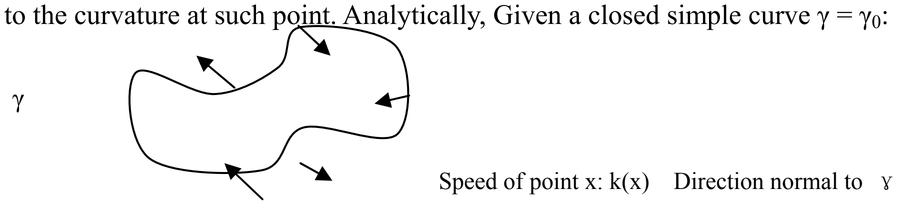

Figure 4 An Example of Curves Flowing by Curvature

$S^1 \to R^2$, we look for a smooth function: $S^1 \times [0; T) \to R^2$ such that for every $\theta \in S^1$ and $t \in [0; T)$. $\vec{N}(\theta; t)$ =

$$\begin{cases} \frac{\partial \gamma}{\partial t}(\theta, t) = k(\theta, t)\vec{N}(\theta, t) \\ \gamma(\cdot, 0) = \gamma_0 \end{cases}$$

“inner” unit normal vector, $k(\theta; t)$ = curvature of $\gamma_t = \gamma(\cdot, t)$ at the point $\gamma(\theta, t)$. This is a nonlinear parabolic system of partial differential equations. There is a theorem “Every smooth, closed, simple curve in the plane during the flow stays smooth, closed and simple. After finite time, it becomes convex, then rounder and rounder and shrinks to a single point in finite time. Rescaling the curve in order to keep the enclosed area constant, it converges to a circle”. This is an example of a nice geometric flow, transforming every element of a family of geometric objects in a canonical representative which is a well known element. Moreover, the deformed object at every time always belongs to the same family hence, the family characteristics are preserved. Even if this flow is possibly the simplest geometric flow, this theorem is definitely nontrivial and its proof requires several ideas and techniques from both analysis and geometry.

3.1 The Ricci Flow by Richard Streit Hamilton

At the end of 70's–beginning of 80's, the study of Ricci and Einstein tensors from an analytic point of view gets a strong interest, for instance, in the static works of Dennis DeTurck. A proposal of analysis of a family of flows, among which the Ricci flow, was suggested by Jean-Pierre Bourguignon. In 1982 Richard Streit Hamilton defines and studies the Ricci flow (equation (1)), that is, the system of PDE's describing the evolution of a metric of a Riemannian manifold. The Ricci flow deforms the metric hence, the local geometry in a selective way: contracting in the directions with positive Ricci tensor and dilating in the ones with negative Ricci tensor. The equation is nonlinear, belonging to the same family of the heat equation. The initial shape of the manifold can be seen as a distribution of curvature, the Ricci flow moves and spreads around such curvature like the heat equation does with the temperature. It is then natural to expect to get asymptotically a uniform distribution that is, a very symmetric geometry, for instance like the one of a sphere. The examples are like Figure 5 and Figure 6. There is the Sphere theorem by Hamilton, which is "If a three-dimensional Riemannian manifold has a positive Ricci tensor, then the normalized Ricci flow deforms it asymptotically in a three-sphere". And first corollary is "If a three-dimensional closed manifold admits a metric with positive Ricci tensor, then it is homeomorphic to the three-sphere". Secondary is "If every three-dimensional, closed and simply connected manifold admits a metric with positive Ricci tensor, then we have a proof of Poincaré conjecture". We have a neckpinch and formation of a cup for bad examples like Figure 7 and Figure 8 as follows.

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}_{g(t)} \tag{1}$$

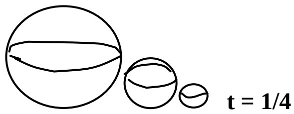

Figure 5 Sphere: $g(t)=(1-4t)/g_0$

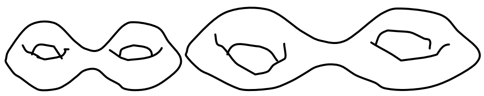

Figure 6 Hyperbolic Surface Constant Curvature-1: $g(t) = (1+4t)g_0$

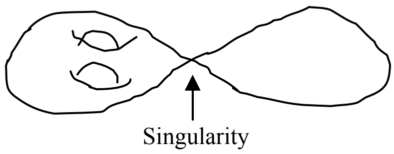

Figure 7 Neckpinch for Bad Example

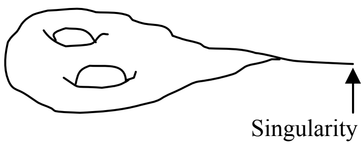

Figure 8 Formation of a Cusp for Bad Examples

3.2 Ricci Flow and Geometry

The Ricci Flow describes a kind of diffusion process which spreads the curvature associated with a Riemannian metric more evenly around a manifold M:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{2}$$

where g_{ij} denotes the Riemannian metric on M and R_{ij} denotes its Ricci curvature. In case of a 3-manifold M, the Ricci curvature completely determines the local geometry of a metric g on M. For example, $R_{ij} = 0$ if and only if g_{ij} is a locally Euclidean metric on M; this has topological consequences too: it means the universal cover of M must be Euclidean 3-space. Under the Ricci Flow the Ricci curvature also spreads around M according to a semilinear diffusion equation. The lecture by Perelman points out a geometric interpretation of Ricci curvature: it measures the initial rate at which the area of a slice of M perpendicular to a direction changes as one parallel translates in that direction. The Ricci Flow may also be viewed as a kind of gradient flow for the metric. We can find more details in the preprint "The entropy formula for the Ricci flow and its geometric applications" by Perelman. Perelman comes to the main application of Ricci Flow to the topology of 3-manifolds: Thurston's Geometrization Conjecture, and its corollary, the Poincare Conjecture. In 1982, Hamilton had already showed that for a 3-manifold of positive Ricci curvature, singularities in the Ricci Flow can be prevented by rescaling. Hence, the problem comes down to understanding how singularities in the Ricci Flow can develop for non-positive Ricci curvature, such as near a pinched neck: This is the type of neck found in the connected sum M # N (Figure 9) of connected 3-manifolds M and N, that is, in the connected 3-manifold obtained by deleting open balls from M and N, and joining these by an $S^2 \times R$ neck. Despite the historical importance of the moment, a few people have dozed off by the time Perelman is ready to present his main theorem: Every closed orientable 3-manifold is a connected sum of pieces, each of which is either $S^2 \times S^1$, S^3/Γ, H^3/Γ, a graph manifold, or a collection of graph manifolds connected with finite-volume H^3/Γ's along incompressible tori. Here Γ denotes the fundamental group of the piece, and each piece of a graph Manifold is Seifert-fibered: Finally, Perelman announces that his main result implies the Poincare conjecture: a simply-connected compact 3-manifold is topologically S^3. The main technique to deal with singularities is geometric surgery. Roughly speaking, this involves slitting each neck and capping-off the slit before the neck can develop into a singularity. The ingredients of the key are the Bishop-Gromov theorem, new estimates on the growth rate of the Ricci curvature, and volume bound.

Figure 9 A Pinched Neck.

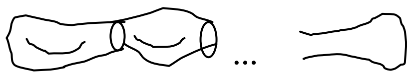

Figure 10 Connected Sum

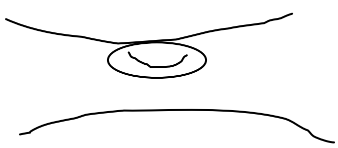

Figure 11 Splitting along an Incompressible Torus

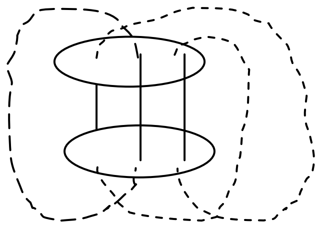

Figure 12 Fundamental Group of the Piece

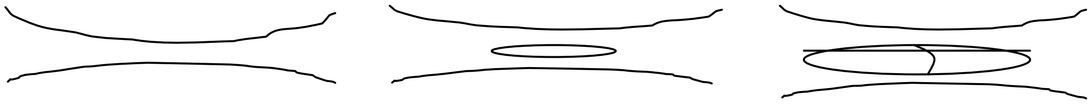

Figure 13 Involves Slitting Each Neck and Capping-off the Slit

4. The Program to Prove Poincaré Conjecture

In the 2-dimensional case, each smooth compact surface can be given a beautiful geometrical structure, as a round sphere in the genus 0 case, as a flat torus in the genus 1 case, and as a surface of constant negative curvature when the genus is 2 or more. A far-reaching conjecture by William Thurston in 1983 claims that something similar is true in dimension 3. His conjecture asserts that every compact orientable 3-dimensional manifold can be cut up along 2-spheres and tori so as to decompose into essentially unique pieces, each of which has a simple geometrical structure. There are eight possible 3-dimensional geometries in Thurston's program. Six of these are now well understood, and there has been a great deal of progress with the geometry of constant negative curvature. However, the eighth geometry, corresponding to constant positive curvature, remains largely untouched. For this geometry, we have the following extension of the Poincare Conjecture. Thurston Conjecture: Every closed 3-manifold with finite fundamental group has a metric of constant positive curvature, and hence is homeomorphic to a quotient S^3/Γ, where $\Gamma \subset SO(4)$ is a finite group of rotations that acts freely on S^3 .The Poincare Conjecture corresponds to the special case where the group $\Gamma \cong \pi_1(M^3)$ is trivial. The possible subgroups $\Gamma \subset SO(4)$ were classified long ago by Hopf, but this conjecture remains wide open. Let us suppose to have a closed, three-dimensional, simply connected manifold, hypothetical counterexample to Poincaré conjecture. The line of proof is as follows. Put a metric on the manifold and deform it with the Ricci flow. Assume that we get a singularity in finite time. If such singularity is like a collapsing three-sphere, then an instant before the collapse our manifold was deformed to a three-sphere contradiction. If the singularity is not a spherical collapse, we try to get the maximum of quantitative information about what is happening to the manifold. To this aim, we need to classify all the possible singularities. With the information of the previous point, we perform a quantitative surgery keeping under control the relevant geometric quantities obtaining one or more new manifolds after the surgery. We restart the Ricci flow on all these new manifolds and repeat the previous steps from the beginning. Show that, assuming the simple connectedness of the initial manifold, after a finite number of steps this procedure ends leaving only a family of collapsing three-spheres. Recovering the initial manifold by inverting the surgery steps, we conclude that the initial manifold actually was a three-sphere too topologically, hence proving Poincaré conjecture. The difficulties are the classification of singularities, the surgery procedure and the Ricci flow with surgery in action. The conjecture of R. Hamilton is “The behavior of the manifold at a singularity can only be one of the three we have seen: collapsing sphere, neckpinch and formation of a cusp.” If the classification conjecture is true, in the neckpinch and cusp cases it is necessary to develop a quantitative surgery procedure which is good enough to allow us to show that in finite time and after a finite number of operations we get a final family of only three-spheres. Despite some positive partial results, the lack of a complete proof of the classification conjecture and of the associated quantitative estimates was an obstacle to have an effective procedure.

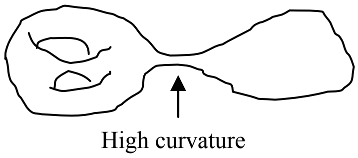

Figure 14 Neckpinch: Before

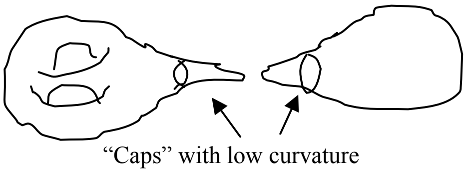

Figure 15 Neckpinch: After

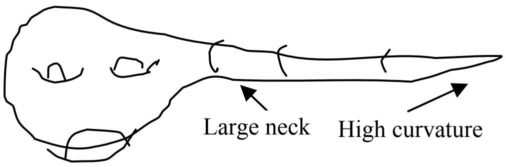

Figure 16 The Surgery Procedure-Cusp: Before

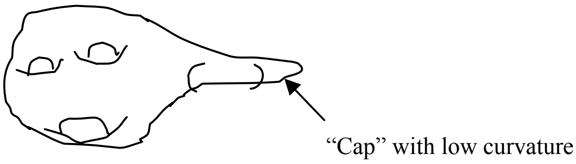

Figure 17 The Surgery Procedure-Cusp: After

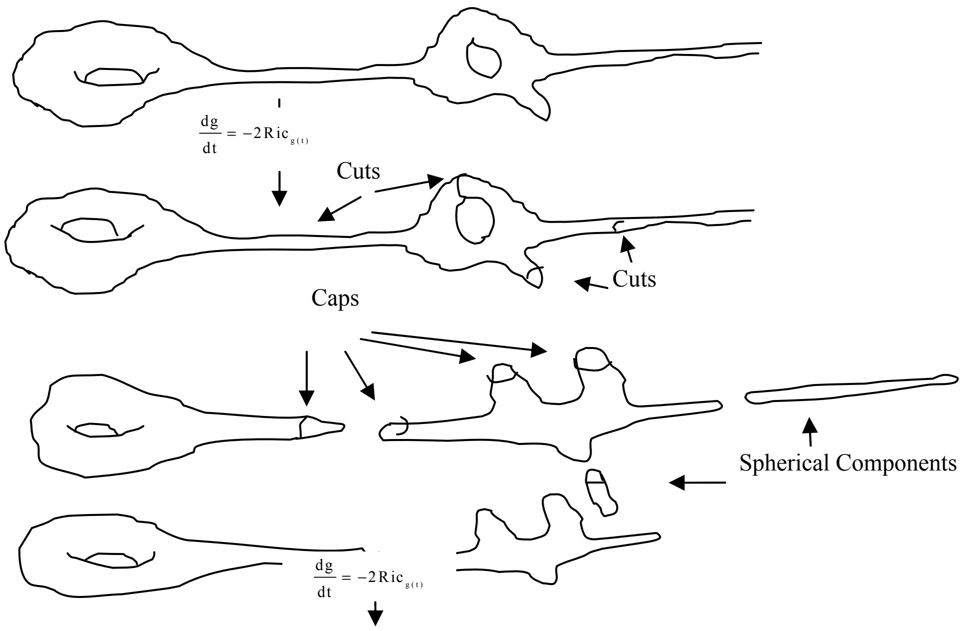

Figure 18 The Ricci Flow with Surgery in Action

5. The Work of Perelman about the Proof of Poincaré Conjecture

In November 2002 Perelman put on the preprint server ArXiv.org the first of a series of three papers, the other two were published in March and July 2003. Vitali Kapovitch sent one email to Perelman for the subject geometrization on 19 November 2002. “Hi, Grisha, Sorry to bother you but a lot of people are asking me about your preprint ‘The entropy formula for the Ricci’. Do I understand it correctly that while you cannot yet do all the steps in the Hamilton program you can do enough so that using some collapsing results you can prove geometrization?” Perelman’s Results has four things. He discovers two new geometric quantities that are monotone during the Ricci flow: the W functional, a sort of entropy, and the reduced length, a kind of distance function in the space-time. Using these, he shows the conjecture of classification of singularities. He finds new estimates on the geometric quantities during the formation of a singularity. He modifies the existing Hamilton’s surgery procedure in order to make it effective and proves that after finite time and a finite number of surgery operations one gets a final finite family of only three-spheres. Bruce Kleiner and John Lott immediately realize the importance of Perelman’s work and start writing down some explicative notes, filling the missing technical details and expanding the most difficult parts and arguments. In June 2006 the Asian Journal of Mathematics publishes, on paper, a work of Zhu Xi-Ping of and Huai-Dong Cao with a complete proof of Poincaré conjecture. In July 2006 John Morgan and Gang Tian publish online on ArXiv, now a standard book, the paper “Ricci Flow and the Poincaré Conjecture” containing a complete and detailed version of Perelman’s proof. This work and the subsequent awarding at the International Congress of Mathematicians in Madrid in August of the same year, of the Fields Medal to Perelman who declined it, mark the formal and substantial acceptation of the mathematical community of his proof of the Poincaré conjecture. Up to now, no mistakes or gaps were found in Perelman’s proof. Moreover, a modified and simplified version in some steps of the proof was presented in 2007 by L. Bessiéres, G. Besson, M. Boileau, S. Maillot and J. Porti. In 2010 the Clay Mathematics Institute awarded Perelman with the Millennium Prize of one million dollars for the proof of the Poincare conjecture. Also this prize was declined by Perelman. Perelman resigned from his position at the Steklov Institute in Saint Petersburg and declared his intention to stop doling mathematics. His three fundamental papers were never published, they are available on the ArXiv preprint server at http://arxiv.org. An even more general conjecture describing the structure of all the three-dimensional manifolds was stated by William P. Thurston. The geomerization conjecture of Thurston is “Every three-dimensional manifold can be cut in geometric pieces”. Only 8 possible geometries: the three with constant curvature and other special five, well known. It implies the Poincaré conjecture, the “space-form conjecture” and the “hyperbolization conjecture”. W. Thurston gave a partial proof. Also this conjecture can be solved by means of the deformation method based on the Ricci flow, like the Poincaré conjecture. Hence, it is natural to ascribe also its proof to the work of Perelman, completed in detail in the papers: Laurent Bessiéres, Gérard Besson, Michel Boileau, Sylvain Maillot and Joan Porti, Geometrisation of three-Manifolds, 2007, John Morgan and Gang Tian, Completion of the Proof of the Geometrization Conjecture, 2008. The 1/4- Pinched Sphere Theorem is “Every Riemannian manifold such that all its sectional curvatures belong to the interval (1 = 4; 1] is diffeomorphic to a sphere” by Heinz Hopf in 1926. It was proved in 2007 by Simon Brendle and Richard Schoen by means of Ricci flow.

6. Conclusion

The Ricci flow was pioneered by Hamilton. But the version of surgery needed for Perelman's argument is extremely delicate as one need to ensure that all the properties of Ricci flow used in the argument also hold for Ricci flow with surgery. Running the surgery in reverse, this establishes the full Thurston geometry conjecture, and in particular the Poincaré conjecture as a special and simpler case. Perelman's Results are as follows. He discover two new geometric quantities that are monotone during the Ricci flow: the W functional and the reduced length. Using these, he shows the conjecture of classification of singularities. He finds new estimates on the geometric quantities during the formation of a singularity. He modifies the exiting Hamilton's surgery procedure in order to make it effective and proves that after finite time and a finite number of surgery operations one gets a final finite family of three-spheres. Running the surgery in reverse, this establishes the full Thurston geometry conjecture, and in particular the Poincaré conjecture as a special and simpler case. Perelman had given the great gift for mathematician all over the world and many new ideas. By no other way could Perelman have attracted more attention to himself, mathematics and the Poincaré conjecture. Now he is just a great mathematician in this century.

References

Bessiéres L., Besson G., Boileau M., Maillot S. and Porti J. (2007). "Weak collapsing and geometrisation of aspherical 3-manifolds", preprint, available online at: http://arxiv.org/abs/0706.2065.

Cao H. D. and Zhu X. P. (2006). "A complete proof of the Poincaré and geometrization conjectures — Application of the Hamilton-Perelman theory of the ricci flow", *Asian Journal of Math*, Vol. 10, pp. 165–492.

Kleiner B. and Lott J. (2006). Notes on Perelman's papers, available online at: http://www.arxiv.org/abs/math.DG/0605667, based on earlier versions posted at http://www.math.lsa.umich.edu/research/ricci flow/perelman.html.

Morgan J. W. and Tian G. (2006). "Ricci flow and the Poincaré conjecture", available online at: http://www.arxiv.org/abs/math.DG/0607607.

Morgan J. W. and Tian G. (2008). "Completion of the proof of the geometrization conjecture", available online at: https://arxiv.org/abs/0809.4040.

Perelmann G. (November 11, 2002). "The entropy formula for the Ricci flow and its geometric applications", preprint, available online at: https://arxiv.org/pdf/math/0211159.pdf.

Perelmann G. (2003). "Ricci flow with surgery on three-manifolds", preprint, available online at: http://arXiv.org/abs/math.DG/0303109 v1.

Perelmann G. (2003, July 17). "Finite extinction time to the solutions to the Ricci flow on certain three-manifolds", preprint, available online at: http://arXiv.org/abs/math.DG/0307245.

Poincaré H. (1906). "Sur la dynamique de l'électron (On the Dynamics of the Electron)", *Rendiconti del Circolo matematico Rendiconti del Circolo di Palermo*, Vol. 21, pp. 129–176.

Richard S. Hamilton. (1982). "Three manifolds with positive Ricci curvature", *Journal of Differential Geometry*, Vol. 17, No. 2, pp. 255–306.