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Right and Left Tail Measurement of Return Distribution in Financial Variables

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Abstract: Cumulative empirical evidence show that distributional information has very important impact on investor's decision making, since the distributional information of asset returns that represents risk and opportunity. In addition, empirical evidence in finance has documented that stock return distributions are not normal. In this paper, we argue that in addition to the left tail information, the right tail distributional information of returns can provide very valuable information to investors and portfolio managers, and the right tail information should be used together with other (say, left tail) information in analyzing financial markets. We consider measures of the right tail distribution. Quantile regression estimators for the right tail measures are developed. The proposed estimation method may also be applied to estimation of other measures in finance.

Key words: distributional information; right and left tail measurement; financial variables
JEL codes: C48, G1

1. Introduction

The distribution or conditional distribution of financial and economic variables is important to investors and decision makers. A widely used model that emphasizes the importance of distributional information, in addition to the mean and conditional mean, is the ARCH/GARCH type model where the role of conditional variance is emphasized.

At present, it is well-known that both the mean and the variance of financial variables can provide important information about financial market. For this reason, most existing models in finance focus on information of the mean and variance (or conditional mean and conditional variance). Under normality assumption, the distribution of return is completely determined by the mean and variance. Using variance as a measure for risk, Markowitz (1952) proposed the mean-variance efficient portfolio which minimizes variance for a given expected return.

Despite the large amount of applications of mean-variance analysis, there is a large amount of empirical studies in finance showing that stock returns are not normally distributed. In addition, it is argued that variance is not a good measure for risk because it is a symmetric measure that penalizes gains and losses in the same way. Although the asset pricing theory of incomplete markets predicts that investors should command higher expected return for bearing higher risks, empirical studies have yielded mixed results on idiosyncratic volatility and there

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has been a lively debate on the role of idiosyncratic volatility in determining cross-sectional stock returns.

Different investors have different preferences on financial securities with different distributions, even when the means and variances are the same across these securities. From a risk management point of view, researchers find that the information contained in the left tail distribution is important in risk management and portfolio construction. Left tail measures such as Expected Shortfall and Value-at-Risk are now widely used in finance applications (e.g., Artzner et al., 1999). Recently, Xiao (2014) argues that, in addition to the left tail information, the right tail of return distribution also contains important information that affects investors' behavior in financial markets. Investors not only look at the left tail distribution of returns to control risk, but also consider the right tail distributional information for opportunity — just like buying insurance as well as lotteries. People try to control risk by purchasing insurance, and they also try to capture opportunities of making a fortune in buying lotteries — although different people may have different attitudes over risk and opportunity, they are interested in the information contained in both the left tail and the right tail of the return distributions. In economics, policy makers need to consider not only the most likely future path for the economy but also the distribution of possible outcomes about that path (Alan Greenspan, 2003).

Distributional information of financial variables, both the left tail and right tail, are important for investment decisions. In the last few decades, researchers have devoted a lot of effort in developing models that can capture decision-makers' behavior more accurately. One important model along this direction is the "cumulative prospect theory" proposed by Tversky and Kahneman's (1992). For an investor characterized by the prospect theory, his/her decision is affected by the whole distribution of return process, not only by the mean and the variance. An investor exhibits particular sensitivity to both the left tail and right tail distribution of the return, and the sensitivity is asymmetric in two tails.

In this paper, we discusses the importance of distributional information and properties of measurements of distributional information. Particular attention is paid to the right tail measurements that summarize the right tail distributional information. Estimators for the right tail measures using quantile regression are introduced.

2. Measures of Right Tail Distribution

To capture right tail information in the distribution of financial variables, appropriate measures are needed. Right tail moments are natural measures capturing the right tail distributional property. Let R be the return of a security, the k-th Right-Tail-Moment Y_k is simply the k-th moment of a gain exceeding a specified upper quantile of the return distribution. Let τ be an upper quantile, say $\tau = 95\%$, if we denote the τ-th quantile of the distribution of R by $Q_R(\tau)$, i.e., $Pr(R<Q_R(\tau))=\tau$, the $(100\tau)\%$ level k-th order Right-Tail-Moment of R is defined as follows:

$$Y_k(\tau) = E[R^k | R \geq Q_R(\tau)]$$

When $k = 1$, we obtain the Right Tail Mean (RTM) $M(\tau) = Y_1(\tau)$. In the case k = 2, we obtain the Right Tail Variance (RTV): Variance of return exceeding the $(100\tau)\%$ level quantile:

$$RTV(\tau) = E\{[R-M(\tau)]^2 | R \geq Q_y(\tau)\}$$

The Right-Tail-Mean (RTM) is an important measure of opportunity. Since it is the right tail counterpart of the Expected Shortfall (ES), we may call it the "Expected Windfall" (EW) as in Wan and Xiao (2009). The Right-Tail-Variance provides important information of good uncertainty of a security.

In many applications, investors look at the *conditional* distribution of returns given available information. In such cases, we consider the *conditional* Right Tail Moments. Let R_t be the return of an asset at time t, and X_t

denotes the vector that contains information available at time t, denote the τ-th conditional quantile of R_t as $Q_{Rt}(\tau|X_t)$, i.e., $\Pr(R_t < Q_{Rt}(\tau| X_t)|X_t) = \tau$, the $(100\tau)\%$ level *k*-th order conditional Right-Tail-Moment of R is defined as:

$$Y_k(\tau,x) = E[R_t^k \mid R_t \geq Q_{Rt}(\tau| X_t, X_t = x]$$

When k = 1, we get the (100τ)% level conditional Right-Tail-Mean of R

$$M(\tau,x) = E[R_t| R_t \geq Q_{Rt}(\tau| X_t, X_t = x]$$

Again, for k > 1, we usually consider the re-centered Right-Tail-Moment. When k = 2, we obtain the conditional Right Tail Variance:

$$RTV(\tau,x) = E\{[R_t - M(\tau,x)]^2 | R_t \geq Q_{Rt}(\tau| X_t, X_t = x\}$$

3. Important Properties of the Right Tail Mean

As a measurement of opportunity, the RTM satisfies some important properties. Let Y be return of an asset, denote the associated Right Tail Mean as μ(Y), then μ(Y) has the following properties:

(1) Monotonicity: For any Y_1, $Y_2 \in X$, if $Y_1 \geq Y_2$, then $\mu(Y_1) \geq \mu(Y_2)$.

(2) Subadditivity: For any Y_1, $Y_2 \in X$, $Y_1+Y_2 \in X$, and $\mu(Y_1+Y_2) \leq \mu(Y_1)+\mu(Y_2)$.

(3) Linear Homogeneity: For any $\lambda \geq 0$, and all $Y \in X$, $\mu(\lambda Y) = \lambda\mu(Y)$.

(4) Translation Invariance: For any $a \in R$, and all $Y \in X$, $\mu(Y+a) = \mu(Y)+a$.

The monotonicity property of RTM indicates that it is consistent with stochastic dominance: if a security’s distribution dominates another security’s, its opportunity measure should be larger — opportunity increases when the return increases. Subadditivity of RTM is a property of no extra synergy. It means that a merger does not bring extra opportunity. In practice, although the optimal level of diversification (measured by the rules of mean-variance portfolio theory) exceeds 300 stocks (e.g., Campbell, Lettau, Malkiel & Xu, 2001; Statman, 2004), it is well-known that the average investor holds much less stocks. This is because that although diversification reduces downside risk, it also reduces the upside opportunity. Investors not only are sensitive to the downside protection, but also care about the upside potential. Such a property is also reflected on the corporate focus in practice. It is found in the literature of merge that marginally profitable projects (whose risks are high) merge to survive a period of distress but, if profitability improves, divesture occurs. Linear Homogeneity says that the opportunity of a financial position grows in a linear way as the size of the position increases. The last property of translation invariance indicates that adding (or subtracting) a sure amount “*a*” to the portfolio simply increase (or decrease) the opportunity measure by “*a*”. The opportunity of a risk-free asset should be the same as the certain payoff provided by the risk-free asset.

4. Estimating Expected Shortfall and Expected Windfall (Left Tail Mean and Right Tail Mean)

Denote the (100τ)% level Expected Shortfall (Left Tail Mean) by S(τ), and the (100τ)% level Expected Windfall (Right Tail Mean) by M(τ). Notice that

$$S(\tau) = E[R|R \leq \alpha(\tau)] = (1/\tau)E[R1(R \leq \alpha(\tau))],$$

and

$$M(\tau) = E[R|R \geq \alpha(\tau)] = (1/(1-\tau))E[R1(R \geq \alpha(\tau))],$$

the quantile regression method provides great convenience in estimating the LTM and RTM. Given a random sample $\{R_t, t = 1,\cdots,T\}$ on Y, we may estimate $\alpha(\tau)$ by the following quantile regression

$$\hat{\alpha}(\tau) = \widehat{argmin} \sum_{t=1}^{T} \rho_\tau(R_t - \alpha)$$

we can estimate the LTM $S(\tau)$ and RTM $M(\tau)$ by replacing the expectations in the above formula by the corresponding sample averages. In particular, the LTM $S(\tau)$ and the RTM $M(\tau)$ can be estimated by

$$\hat{S}(\tau) = \frac{1}{\tau T}\sum_{i=1}^{T} R_t 1(R_t \leq \hat{\alpha}\,(\tau))$$

$$\widehat{M}(\tau) = \frac{1}{(1-\tau)T}\sum_{i=1}^{T} R_t 1(R_t \geq \hat{\alpha}\,(\tau))$$

The RTM (Expected Windfall) exists provided $E|Y|<\infty$. But if we further assume that $E|R|^k<\infty$ for $k \geq 2$, under appropriate regularity assumptions, the estimated RTM is a root-T consistent estimator, and is asymptotically normal. We first consider this case when the second moment exists.

For convenience of asymptotic analysis, we first give some regularity conditions that are sufficient for the consistency and asymptotic normality of the proposed estimators, although they might not be the weakest possible.

Assumption A: $\{R_t\}$ is a strong mixing stationary process with mixing coefficient $\alpha(n)$ satisfying $|\alpha(n)| < C\rho^n$ for some $C > 0$, and $0 < \rho < 1$, $E\|R_t\|^{2(1+\delta)} < \infty$, for some $\delta > 0$.

Assumption A assumes that the process is weakly dependent and appropriate LLN and CLT applies to sums of functions of these random variables. Under these assumptions, the limiting distribution of $S(\tau)$ and $M(\tau)$ are given by

$$\sqrt{T}(\hat{S}(\tau) - S(\tau)) \Rightarrow N(0, \omega_U^2(\tau))$$

$$\sqrt{T}\left(\widehat{M}(\tau) - M(\tau)\right) \Rightarrow N\left(0, \omega_V^2(\tau)\right)$$

Where $\omega_U^2(\tau)$ and $\omega_V^2(\tau)$ are the long-run variances of U_t and V_t, i.e., $\omega_U^2(\tau) = \sum_{h=-\infty}^{\infty} COV(U_t, U_{t+h})$, $\omega_V^2(\tau) = \sum_{h=-\infty}^{\infty} COV(V_t, V_{t+h})$.

The estimators are consistent and asymptotically normal at rate square root of sample size. Furthermore, the estimation of the τ-th quantile of R does not appear to affect the limiting distribution of the estimators of RTM.

5. Tail Measures in the Presence of Heavy-tail Distributions

The concept of left or right tail measures, ES or RTM, can be applied to many other economic or financial variables. In some applications, the distribution of time series has heavy tails and variance may not exist. For example, electricity prices can be subject to large spikes due to supply/demand imbalances that cannot be temporally mediated (Weron, 2008). There is also evidence on heavy-tail behavior about the distribution of wealth and income. Klass et al. (2006) and Nirei and Sonma (2007) documented that the tail exponent of wealth is around 1.5. Gopikrishnan et al. (2000) find that trading volumes for the 1,000 largest U.S. stocks have Pareto tail with exponent around 3/2. The 1987 crash delivered daily a return on the broad S&P500 index that was over 20 standard deviations below the mean. More recently, the "flash crash" of May 6th, 2010, showed how far stock prices could move in a very short period of time. At 2:42 pm, with the Dow Jones Industrial Average down more than 300 points for the day, the index began to fall rapidly, dropping more than 600 points in 5 minutes for an almost 1000 point loss (or about nine percent) on the day by 2:47 pm. Twenty minutes later, by 3:07 pm, the market had regained most of the 600 point drop. We note that a lot of financial theory concerning diversification and the risk return trade-off does not require a finite variance. For example the CAPM, "Mean Variance

Efficiency", and "Two fund separation" are known to hold for the more general class of elliptical distributions that are characterized by a location vector μ and a scale matrix Ω. The scale matrix Ω = (ω_{ij}) need not be a covariance matrix. Press (1982) shows that provided the expected return exists and is finite ER – r_f= $\beta_{CAPM}(R_m - r_f)$, $\beta_{CAPM}\}$ = $(\omega_{im})/(\omega_{mm})$, see also Fama and Miller (1972) and Samuelson (1967). So, we don't need a variance. Stock and Watson (2007) discussed the 1987 Black Friday effect on the Dow Jones and its implications for non-normality. Yet many risk measures are constructed from variance, and so may be non-robust to large movements in series. The consequences of large crashes are enormous, and it is important to have risk management tools that reflect this possibility and are robust to it.

Suppose that Y is the financial variable that we are interested. In this section, we consider the case where the RTM is defined (E|Y| < ∞) but the variance of Y is infinite, and so the above results do not necessarily hold. In such cases, the ES or RTM can still be consistently estimated, but limiting distributions are different and statistical inference requires different methods.

Given a sample $\{Y_1,\ldots,Y_T\}$, the τ-th quantile of Y, can be estimated simply by the τ-th sample quantile of, say, $\hat{\alpha}(\tau)$. Denote a specified lower (left) quantile by $\underline{\tau}$ (say $\underline{\tau}$ = 5%) and a upper (right) quantile by $\bar{\tau}$(say $\bar{\tau}$ = 95%), we may construct the following estimators of left tail mean (i.e., the Expected Shortfall) S($\underline{\tau}$), and the right tail mean M($\bar{\tau}$):

$$\hat{S}(\underline{\tau}) = \frac{1}{\tau T}\sum_{i=1}^{T} Y_t 1(Y_t \leq \hat{\alpha}\,(\underline{\tau}))$$
$$\widehat{M}(\bar{\tau}) = \frac{1}{\tau T}\sum_{i=1}^{T} Y_t 1(Y_t \geq \hat{\alpha}\,(\bar{\tau}))$$

Since the asymptotic behavior of $\hat{S}(\underline{\tau})$ and $\widehat{M}(\bar{\tau})$ are very similar. We present the asymptotic distribution of $\hat{S}(\underline{\tau})$below.

Under the assumptions that (Y_t) are realizations from a strictly stationary sequence with regularly varying tail probabilities with tail thickness index θ, and is strongly mixing with geometrically declining mixing coefficients,

$$T^{(\theta-1)/\theta}(\hat{S}(\underline{\tau}) - S(\underline{\tau})) \Rightarrow \frac{1}{\tau}\underline{S}$$
$$T^{(\theta-1)/\theta}(\widehat{M}(\bar{\tau}) - M(\bar{\tau})) \Rightarrow \frac{1}{\tau}\bar{S}$$

Where $\underline{S}$ and $\bar{S}$ are stable distributions.

Under heavy tailed assumptions, the estimators are consistent at a rate depending on the tail thickness parameter and have a stable limiting distribution; estimation of the quantiles does not affect the limiting distribution. However, the limiting distribution is very complex and depends on the dependence properties of the data as well as on the tail thickness parameter, so that "plug-in inference" is very complicated.

In order to conduct statistical inference based on the proposed estimators, we need to estimate the asymptotic distributions somehow. In the case of finite variance time series, the ES estimator is root-T consistent and asymptotic normal with a variance that can be consistently estimated. In the case with infinite variance, to conduct inference about S($\underline{\tau}$) and M($\bar{\tau}$), we need to estimate consistently the parameters (θ,c_+,c_-) under the weak conditions we have imposed, which is a difficult task. The parameter θ can be estimated consistently under weak dependence conditions by many methods, see for example Hill (2010). Estimation of the spectral measure has been investigated for i.i.d data, see for example Einmahl, de Haan, and Piterbarg (2001). But these results do not cover estimation of c_+,c_- under weak dependence conditions. In this paper, we propose a general method based on subsampling (Politis & Romano, 1999). This method is also consistent when the variance of the series exists and so is robust with regard to the tail thickness parameter. Since the analysis of S($\underline{\tau}$) and M($\bar{\tau}$) are very similar. We present the discussion of S($\underline{\tau}$) below.

Let $\widehat{\theta}$ be a consistent estimator of θ, given the random sample $\{Y_t, t=1,\ldots,T\}$, we consider subsamples of size M

$$\{Y_t,\ldots,Y_{t+M-1}\},\ t=1,\ldots,T-M+1,$$

and estimate the expected shortfall based on subsamples. Thus the estimators are

$$\hat{S}(\underline{\tau}, M, t) = \frac{1}{\tau M}\sum_{s=0}^{M-1} Y_{t+s} 1(Y_{t+s} \leq \widehat{\alpha_t}(\underline{\tau}))$$

Where $\widehat{\alpha_t}(\tau)$ is the corresponding estimators based on the subsample $\{Y_t,\ldots,Y_{t+M-1}\}$.

We approximate the sampling distribution of $T^{(\theta-1)/\theta}(\hat{S}(\underline{\tau})$ - $S(\underline{\tau})$),denoted by$\widehat{F_T}$ (y), by

$$\widehat{F_{T,M}}(y) = \frac{1}{T-M+1}\sum_{i=1}^{T-M+1} 1(M^{(\widehat{\theta}-1)/\widehat{\theta}}[\hat{S}(\underline{\tau}, M, t) - \hat{S}(\underline{\tau})] \leq y)$$

Let F(y) be the limiting distribution function of $T^{(\theta-1)/\theta)}(\hat{S}(\underline{\tau})$-S(τ)). Under the assumptions that the tail index θ is estimated by $\widehat{\theta}$ at rate faster than logT in the sense that log(T)($\widehat{\theta}$ -θ)→0, and M→∞, and M/T→0, as T→∞,

$$F_{T,M}(y) \to F(y)$$

The subsampling method is "robust" in the sense that it is also consistent even in the case of finite variance, where normal asymptotics prevail.

6. Conclusion

The distributions of financial return processes contain rich information that affects the investors' decision. Two important distributional measures are the left tail mean and the right tail mean. Both the left and right tail measures can be estimated by the sample analogues. There is no doubt that right tail information, together with the left tail information, have important impacts on financial decisions such as portfolio construction.

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