Journal of Business and Economics, ISSN 2155-7950, USA
January 2015, Volume 6, No. 1, pp. 13-17
DOI: 10.15341/jbe(2155-7950)/01.06.2015/002

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Rational Interplay: A Philosophical Critique and Revision of Nash Equilibrium

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Abstract: Rational interplay does not always equilibrate, even assuming complete information, common knowledge, and a unique pure-strategy Nash equilibrium. One reason is that players sometimes are indifferent between equilibrium and other strategies. Another is that a fine enough division of opportunity into distinct strategies can efface equilibria that coarser divisions reveal. A revised equilibrium concept escapes these problems.

Key words: game theory; Nash equilibrium
JEL code: C7

If rational actors aim to maximize utility, to make the best choices they can, then rational interactors should aim to choose best responses to each other's choices. Such is the obvious rationale for predicting Nash equilibria, game outcomes from which no solo deviation would be profitable. It is fallacious: rational players can fail to produce a Nash equilibrium even when circumstances make it dead easy to produce one. Some obstacles to Nash equilibration are well known, even obvious: sometimes players are not fully rational, or they do not have enough information to choose best responses to each other's choices, or there is no way for them all to do so (no Nash equilibrium), or there are too many ways, each requiring coordination. But even in the presence of complete information, common knowledge, and a unique chance-free Nash equilibrium, the outcome of rational interplay is not always a Nash equilibrium. It is not that equilibrium strategies are irrational in the problem cases, but other strategies are no less rational and sometimes more compelling. I should emphasize that my critique is philosophical: predictive success aside, the definition of Nash equilibrium does not capture the concept of rational interplay. A revision does better.

1. Games

In a game, *players* 1, 2, …, n simultaneously choose *strategies* from sets $S_1, \ldots, S_n$ respectively, say x_i from S_i, and each player i receives *utility* or *payoff* $fu_i(\boldsymbol{x})$ from *outcome* $\boldsymbol{x} = (x_1, \ldots, x_n)$. To predict play I assume three things about players:

- *Rationality*. A player's sole objective is the greatest possible payoff for himself.

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- *Complete information*. Every player knows the whole game—players, strategy sets, payoffs.
- *Common knowledge*. Every player knows the previous two things, knows that every player knows them, and so on.

2. Nash Equilibria

Referring to players as *i*, *j*, etc. and outcomes as $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$, etc., define:

- A *Nash equilibrium* is an outcome **x** such that $u_i(\boldsymbol{x}) \geq u_i(\boldsymbol{y})$ for every *i* and every *i*-variant $\boldsymbol{y}$ of $\boldsymbol{x}$ (every $\boldsymbol{y}$ in which $y_j = x_j$ whenever $j \neq i$).
- An *equilibrium strategy* is a component of some Nash equilibrium.

In the five little games of Figure 1, Rose chooses *U* (up) or *D* (down), Colin *L* (left) or *R* (right). Outcomes (*UL*, *UR*, *DL*, *DR*) match tabular cells containing payoffs, Rose's before Colin's. Nash equilibria are marked*. So Game 1 (Prisoners' Dilemma) has one Nash equilibrium, Game 2 none, and Games 3-5 two each. If both players choose equilibrium strategies, the outcome has to be a Nash equilibrium in Game 3 but not 4 or 5, where to equilibrate they must somehow *coordinate*, or aim at the same equilibrium.

Game 1	*L*	*R*
U	3,3	1,4
D	4,1	2,2*

Game 2	*L*	*R*
U	3,1	2,4
D	1,4	3,2

Game 3	*L*	*R*
U	3,2*	4,2*
D	1,4	2,3

Game 4	*L*	*R*
U	2,2*	0,0
D	0,0	2,2*

Game 5	*L*	*R*
U	4,4*	1,3
D	3,1	2,2*

Figure 1 Five Little Games

Sometimes game structure helps them do that, not in Game 4 but in Game 5 (Stag Hunt), where one equilibrium (*UL*) is *e-dominant*, unanimously preferred to the other(s). In extensive-form games, strategies are decomposed into successive moves, conditional on earlier moves by all players, and that extra structure sometimes reveals equilibria that entail foolish moves. An academic industry is devoted to refining out such unlikely targets of coordination. Here I shall mostly ignore multiple equilibria and wholly ignore extensive forms, not to disparage problems of coordination and refinement but to highlight new problems.

3. Mixed Strategies

Game 2 has no Nash equilibrium, but we can give it one by expanding opportunities to include *mixed strategies*. For Rose a mixed strategy has the form "*U* with probability *p*, *D* with probability 1-*p*", and for Colin, "*L* with probability *q*, *R* with 1-*q*". To equilibrate, Colin must pick q so that *U* and *D* have the same expected utility for Rose—so that $3q + 2(1-q) = q + 3(1-q)$. Rose must likewise pick p so $p + 4(1-p) = 4p + 2(1-p)$. Solving for *q* and *p* we have *q* = 1/3 and *p* = 2/5. By equalizing the expected utilities of *U* and *D*, Colin has given not only *U* and *D* but all mixed strategies formed from them the same expected utility ($2\frac{1}{3}$), as has Rose ($2\frac{4}{5}$) for Colin's strategies. Therefore, neither player can gain from any solo change of strategy: their mixed strategies are in equilibrium. If instead *U* and *D* had unequal expected utilities, say *U*'s greater, then Rose's best choice would be plain *U*, to which Colin's best response would be *R*, to which Rose's would be *D*, and so on: no equilibrium.

An obvious problem is how to effect mixed strategies. An old “solution” assumes that the focal game is played over and over. Then Rose can choose *U* two times out of five while Colin chooses *L* one time out of three. But the repeated play of Game 2 is a vastly different game from Game 2. For Game 2 itself a mixed strategy requires some exotic mental exercise. Is it feasible? Grant that it is.

A deeper problem remains. Given Rose’s equilibrating choice of $p = 2/5$, all strategies open to Colin, pure and mixed, have the same expected utility, $2\frac{4}{5}$. Because his sole objective is the greatest possible payoff, Colin is equally willing to choose any of them. His equilibrium strategy is but a point in a continuum of equally good strategies. He has no reason to favor that point, and he would have to be nuts to work at finding it. Not only have players no reason to equilibrate but it is quite certain they would not.

4. Unique Pure-Strategy Equilibria

In the two games of Figure 2, Nash equilibria exist, are unique, and consist of pure strategies. Even so, rational interplay need not equilibrate. In Game 6 let Rose choose *U*, her equilibrium strategy. Why should Colin choose *L*, his own equilibrium strategy? It does maximize his payoff, but *R* does too. Because his sole objective is the greatest possible payoff, should he not be as willing to choose *R* as *L*?

Game 6		
	L	*R*
U	1,1*	0,1
D	0,1	1,0

Game 7			
	L	*C*	*R*
U	1,1*	1,0	0,1
D	0,-1	0,1	1,0

Figure 2 Problems with Unique Pure-strategy Nash Equilibria

Yes, *L* (weakly) *dominates R*: it is at least as good for Colin in both rows and better in one. Its advantage over *R* is as a hedge, in case Rose does not choose her own equilibrium strategy. But Game 7 (with *C* for *center*) reverses that advantage: *R* now dominates *L*. If being dominated does not discredit a strategy then it does not rule out *R* in Game 6. If it does then it rules out *L*, Colin’s sole equilibrium strategy, in Game 7. There, if he seeks a hedge against nonequilibrium behavior by Rose, Colin should favor *R*. In any case, he has no reason at all to choose his equilibrium strategy.

Dominated equilibrium strategies are old news. But in the story beneath that headline, the successive elimination of dominated strategies refines a multiplicity of Nash equilibria down to one. In Game 7 it “refines” one down to none.

From his own point of view, Colin’s position in Games 6 and 7 is the same as in Game 3, where he is equally willing to choose *L* or *R* (“interchangeable” equilibrium strategies), given Rose’s choice of *U*. True, *R* is not an equilibrium strategy in Game 6 or 7. But from Colin’s point of view that is irrelevant: his goal is maximum payoff, not equilibration.

5. Strict Nash Equilibria

In Figure 2 the Nash equilibria are not *strict*: Colin likes *UL* as much as *UR* but no more. Not that all reasonable equilibria are strict: those of Game 3 are not. But strictness rules out some pathological cases.

However, rational players might bypass even strict—and unique—Nash equilibria. In Game 8 of Figure 3 (with *M* for *middle*), *DR* is the sole Nash equilibrium, and it is strict. But our three assumptions virtually ensure that Rose and Colin would bypass *DR* in favor of one of the four outcomes—any one of them—in the upper left. That would give them greater payoffs, both of them know it, both know that both know it, and once there, neither would switch to *D* or *R*. None of those four outcomes is a Nash equilibrium, but equilibration is no one's goal.

Game 8

	L	*C*	*R*
U	1,0	0,1	-2,-2
M	0,1	1,0	-2,-2
D	-2,-2	-2,-2	-1,-1*

Game 9

	L-or-*C*	*R*
U-or-*M*	h,k*	-2,-2
D	-2,-2	-1,-1*

where $0 \le h \le 1$, $0 \le k \le 1$

Figure 3 Problems with Strict Nash Equilibria

I have redrawn Game 8 as Game 9 by fusing two of the three strategies open to each player into one "either-or" strategy. If Game 8 is a correct description of some situation then Game 9 is an equally correct if less specific description of the same situation. And unlike Game 8, Game 9 has two Nash equilibria, one of them e-dominant. That game is tantamount to Game 5, Stag Hunt, where players are near certain to coordinate around the e-dominant equilibrium, rejecting *DR*. Therefore, back in Game 8, they are near certain to produce one of the four upper-left outcomes rather than equilibrate.

If factual accuracy is our goal it is quite arbitrary how we slice a player's loaf of opportunity into distinct strategies. We are as free to say that Rose has two strategies as three. If I say simply that you can buy a car or not, I leave out makes and models. If I decompose your "buy" option into a list of makes and models, I omit price, color, time of purchase, and much more. Of it I start with a long list of options (including ways of not buying a car, such as stealing one), I might find it convenient or revealing to simplify, as Game 9 does for Game 8, by fusing several options into one "either-or" option. There is no right way to do any of that: all ways are equally faithful to the facts. But the existence and identity of Nash equilibria depend on how finely or coarsely we divide opportunity—we analysts, that is, not the players themselves.

6. Revision

So the less problematic Nash equilibria are strict, and even they can fail to capture rational interplay when strategic options are divided too finely. In §5 we found a second, more attractive equilibrium after coarsening the original division by fusing some strategies into fewer, less specific ones. The fused strategies of Game 9 can be represented by sets: *U*-or-*M* is now the set $\{U, M\}$; to choose it is to choose *U* or *M*. In general let player *i*'s strategic options be all the nonempty subsets of S_i. Because that includes unit sets, the finest division is pooled with all the coarser ones: no options are left out.

To avoid a proliferation of jargon I shall use old words for new things but append superscript *S* (for *set*). So a *strategy*^{S} is a nonempty subset of some S_i, and an *outcome*^{S} is a vector $(A_1, \ldots, A_n)$ of nonempty subsets of $S_1, \ldots, S_n$ respectively. Referring to outcomes^S as $\mathbf{A} = (A_1, \ldots, A_n)$, $\mathbf{B} = (B_1, \ldots, B_n)$, etc., define:

- $\prod \mathbf{A}$ = the Cartesian product $A_1 \times \cdots \times A_n$.

- An *equilibrium*^{S} is an outcome^S**A** such that $u_i(\mathbf{x}) > u_i(\mathbf{y})$ for every i, every $\boldsymbol{x}$ in $\prod \mathbf{A}$, and every i-variant $\boldsymbol{y}$ of $\boldsymbol{x}$ with $y_i \notin A_i$.

What makes $\boldsymbol{A}$ an equilibrium^S, understood in a strict sense, is that any solo change of strategy^S from $\boldsymbol{A}$ would be costly to the player who changed. But players choose strategies^S by choosing individual strategies belonging to them, and cost is measured by utility functions defined for individual outcomes, not outcomes^S. So $\boldsymbol{A}$ is an equilibrium^S if no player (i) can make a solo change of strategy from any individual outcome compatible with $\boldsymbol{A}$ (any $\boldsymbol{x}$ in $\prod \mathbf{A}$) to one that is not (to an i-variant $\boldsymbol{y}$ of $\boldsymbol{x}$ with $y_i \notin A_i$) without incurring a utility loss ($u_i(\boldsymbol{y}) < u_i(\boldsymbol{x})$).

In Game 8, for example, $(\{U, M\}, \{L, C\})$ is an equilibrium^S, as expected. The individual outcomes compatible with it are *UL*, *ML*, *UC*, and *MC* (a.k.a. (U, L), (M, L), etc.). As the definition requires, Rose prefers *UL* and *ML* to their individual Rose-variant, *DL*, and likewise *UC* and *MC* to *DC*.

Game 8 has two more equilibria^S. One is $(\{U, M, D\}, \{L, C, R\})$, as in general is $\boldsymbol{S} = (S_1, \ldots, S_n)$ in every game although it makes the vacuous prediction that something or other will happen. We can simply note that some equilibria^{S}make stronger predictions than others, or we can define:

- A *strong equilibrium*^{S} is an equilibrium^S**A** such that for no equilibrium^S**B** is $\prod \mathbf{B}$ a proper subset of $\prod \mathbf{A}$.

In Game 8 only $(\{D\}, \{R\})$ and $(\{U, M\}, \{L, C\})$ are strong equilibria^S, matching the two Nash equilibria of Game 9. But one of the latter is e-dominant, suggesting:

- An *e-dominant equilibrium*^{S} is a strong equilibrium^S$\boldsymbol{A}$ such that, for every other strong equilibrium^S$\boldsymbol{B}$, $u_i(\boldsymbol{x}) > u_i(\boldsymbol{y})$ for all i, all $\boldsymbol{x}$ in $\prod \mathbf{A}$, and all $\boldsymbol{y}$ in $\prod \mathbf{B}$.

Only $(\{U, M\}, \{L, C\})$ is an e-dominant equilibrium^S of Game 8.

In Game 1 the Nash equilibrium is *DR*, the strong equilibrium^S $(\{D\}, \{R\})$—no real change. Similarly, Games 4 and 5 both have strong equilibria^S $(\{U\}, \{L\})$ and $(\{D\}, \{R\})$, the former e-dominant in Game 5. Game 2 has no Nash equilibrium and 6 and 7 only worthless *UL*. All three now have the strong but vacuous equilibrium^S**S**: there is nothing to predict. But in Game 3 the two Nash equilibria, *UL* and *UR*, are both reasonable. Because they are not strict, $(\{U\}, \{L\})$ and $(\{U\}, \{R\})$ are not equilibria^S. But $(\{U\}, \{L, R\})$ is a strong equilibrium^S that predicts the same behavior.

Five facts merit notice: (1) Vacuous $\boldsymbol{S}$ is always an equilibrium^S. (2) Every game also has at least one *strong* equilibrium^S, an equilibrium^S$\boldsymbol{A}$ with ⊆- least $\prod \mathbf{A}$. But (3) it need not have a unique or e-dominant one, as witness Game 4. (4) Thanks to our three assumptions, when an e-dominant equilibrium^S exists it is a good bet, a reasonable thing to predict: every player sees its mutual advantage and how to get it and knows that the others do too and that they know that he does. By contrast, (5) an e-dominant *Nash* equilibrium is not always a good bet: it is in Game 5 (or 9) but not in Game 8, where *DR* is the unique and, therefore, e-dominant Nash equilibrium but is mutually costly compared with equilibrium^S $(\{U, M\}, \{L, C\})$.

7. Conclusion

The old problems with Nash equilibria are ones of dearth and glut, no equilibria or too many. My new ones lie in between, where Nash equilibria exist and are unique. In the problem cases, rational interplay does not always block them, but neither does it ever compel them. One reason is that players sometimes are indifferent between equilibrium and other strategies. Another is that fine enough divisions of strategic opportunity can efface Nash equilibria. Coarser divisions, got by fusing strategies, can reveal more equilibria (or equilibria^S), sometimes e-dominant ones.