Journal of Business and Economics, ISSN 2155-7950, USA
July 2014, Volume 5, No. 7, pp. 1029-1051
DOI: 10.15341/jbe(2155-7950)/07.05.2014/009

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Empirical Modeling of Direct Real Estate *Ex Ante* Systematic Risk and Total Risk Behavior under the Duration Risk, Time-varying Risk and Garch Risk

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Abstract: Based on well-established financial theories like duration, convexity, the CAPM and the real estate equivalent yield model, this paper investigates a model for the *ex ante* measurement of the direct real estate systematic risk and direct real estate total risk, in terms of the non-linear exposure to movements in the direct real estate yield. Empirical validation is conducted to estimate the direct real estate duration beta and the time-varying beta, within the context of Singapore's real estate market that comprises the luxury residential, prime office and retail sectors. With the aim to estimate the direct real estate total risk, this paper restructures the resulting and *ex ante* direct real estate modified duration model and it assesses the measurement in comparison with a GARCH (generalized autoregressive conditional heterogeneity) risk model. The *ex ante* direct real estate modified duration model has the potential to measure the real estate systematic and total risk in an expectation form. The test of duration beta against the time-varying betas reveals that the time-varying betas tend to be over-stated. It reveals that the luxury residential sector and prime office sector are inclined to move in opposite direction in terms of both the duration beta and time-varying beta. This has significant meaning for real estate investors while making decisions on asset allocation and portfolio management.

Key words: direct real estate systematic risk; total risk; real estate duration beta; time-varying beta; GARCH risk model and Singapore real estate market

JEL code: R1

1. Introduction

Traditionally, financial risk is defined as the dispersion of unexpected outcomes owing to movements in financial variables. Financial risk is often measured by the volatility, variance or standard deviation (σ) of the asset return. In this regard, both the positive and negative deviations from the asset return's mean are viewed as sources of risk. Financial risks are generated by the movements of financial factors, such as interest rate, exchange rate and the underlying commodity price, that are respectively denoted as interest rate risk, exchange rate risk and

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commodity risk. In this sense, the definition of risk can be better understood through defining the variables of interest such as portfolio value, earnings, capital expenditure or a particular cash flow.

Thus, the measurements of the exposure to movements in the underlying variables can be expressed in several versions. In the fixed-income asset market and the stock market, the linear exposure to the movements in interest rates is defined as the duration and the systematic risk (β) respectively. In the derivative market, the exposure to movements in the value of the underlying asset is defined as the delta (δ). The second-order or quadratic exposure to a financial variable is defined as convexity in the fixed income market and gamma (γ) in the derivatives market. So, convexity measures the change in duration resulting from the change in interest rate, while gamma (γ) measures the change in delta (δ) as the value of the underlying asset in the derivative market changes.

To a large extent, direct real estate asset investment like other investments, is required by investors to achieve a stream of expected income flows, say, the net operating income. Real estate risk exists owing to the uncertainties of these expected income flows. The measurement of real estate risk, in conformity with modern portfolio theory, should reflect an investor's *ex ante* expectations rather than focus on what has happened in the past. Historic measures of risk are merely helpful in forecasting expected risk under set scenarios. In modeling real estate risk, it should not be measured in function of what happened (i.e., actual past volatility) but in function of what might have happened and its probability.

Owing to limited empirical data availability, statistical techniques such as sensitivity analyses, probability analysis and Monte Carlo simulation are used in real estate risk assessment to overcome this problem. Real estate structural risk factors can be further investigated via these methods and this is of much significance since the real estate risk factors in turn affect the portfolio return, say, the Internal Rate of Return (IRR). By controlling the other risk factors, subsequent investigation of a certain key real estate risk factor would surely shed light on the risk behavior of the real estate portfolio return, and should give the investor better insight in order to adjust his exposure to the volatility of the structural real estate risk factors that in turn are the real estate risk sources.

Hence, this paper introduces an augmented duration model to measure the direct real estate systematic risk and the direct real estate total risk, in the form of the non-linear exposure to movements in the direct real estate yields at the individual asset, sector or market levels through an *ex ante* but modified fixed-income duration model, in combination with the direct real estate equivalent-yield valuation model. In this model, limited information is being provided through the lease structure of a direct real estate asset. In other words, this paper investigates the potential modification of the fixed-income duration to measure the return volatility of a direct real estate asset (for example, a complete office building) or a real estate sector relative to the wider real estate market, and then to measure the corresponding total risk of that real estate asset or sector.

In addition, this paper models the estimation of the direct real estate total risk, on the basis of a reliable quarterly data set, where the augmented duration model is formulated as a non-stochastic model principally from a conventional freehold, term and reversion valuation model. The required data set for this paper is obtained from the Jones Lang LaSalle Real Estate Information Service-Asia (JLL REIS-Asia). It is tested for normality and stationarity to assess its appropriateness as a specific investment asset class research index for each of the three sectors of the Singapore real estate market—the luxury residential sector, prime office sector and prime retail sector.

Based on the model of direct real estate systematic risk and total risk under the *ex ante* duration risk, time-varying risk and GARCH risk, this paper further utilizes the JLL REIS-Asia data set for the luxury residential, prime office and prime retail sectors that constitute the Singapore real estate market to measure the systematic risk and total risk. The paper then carries out the direct real estate beta (systematic risk) model estimations via the direct real estate sector duration beta and the corresponding time-varying beta regression. Subsequently, the paper discusses the measurement of the direct real estate asset total risk under the direct real estate duration risk and the GARCH risk models while the final part of the paper concludes the findings.

2. The Data Set

The Jones Lang Lasalle Real Estate Intelligence Service-Asia[1] (JLL REIS-Asia) data set is obtained for this paper, and it is analyzed for partial or complete normality behavior. If the data set is found to be partial normally distributed and thus time variant, it may well be appropriate to investigate the time-varying nature of the return volatility of a real estate asset (or sector), relative to the market subsequently. The associated real estate capital values (CV_s), initial yields (IY_s) and net effective rents on net leasable area, are to be tested in order to establish whether they deviate much from the normal distribution, and therefore from constant variance. The normality test is conducted for the period between 1989 and 2001.

The *CV* and rental values are measured on the basis of thirty buildings from each of the three different prime real estate sectors for Singapore. These sectors comprise the prime office sector in the Raffles Place central business district (CBD), the luxury residential sector and the prime retail sector. *CVs* are measured in terms of Singapore dollars (S$) per sqm, the net effective rents in terms of S$ per sqm on net leasable area and the initial yields in terms of annualized percentages.

Results of the normality tests are presented in Table 1. These tests consist of the ratio of the skewness to standard deviation (*SD*), the ratio of the kurtosis to *SD* and the Jarque-Bera (*JB)* test. Normality is observed when the skewness-to-*SD* ratio and the kurtosis-to-*SD* ratio fall between -2 and +2. If the *JB* test is found to exceed the critical value at a particular significance level (i.e., 1%, 3% or 5%) of the chi-squared distribution, and with two degrees of freedom, then the hypothesis that a variable of interest is normally distributed, is rejected.

Table 1 Statistical Analysis of Prime Real Estate Sectors, Singapore

	CV OFF	*CV RES*	*CV RET*	*IY RES*	*IY RET*	*IY OFF*	*NR RES*	*NR RET*	*ER OFF*	*NR OFF*
Std. Dev.	4515.8	2665.0	2914.8	0.9770	0.950	0.9027	61.741	498.0993	178.61	192.62
Skewness	0.7081	-0.2489	-0.3046	0.9325	0.833	0.8440	-0.396	0.4484	-0.205	-0.193
Skewness/Std. Dev.	0.0002	-0.0001	-0.0001	0.9544	0.876	0.935	-0.006	0.0009	-0.001	-0.001
Kurtosis	1.9894	1.5357	2.1010	2.3005	2.288	2.7114	2.0647	1.9545	1.9399	1.762
Kurtosis/Std. Dev.	0.0004	0.0006	0.0007	2.3546	2.406	3.0037	0.0334	0.0039	0.0109	0.009
Jarque- Bera	5.5498	4.3851	2.1622	7.2736	6.017	5.3770	2.7523	3.4783	2.3675	3.083
Probability	0.0624	0.1116	0.3392	0.0263	0.049	0.0680	0.2525	0.1757	0.3061	0.214

Source: Author; JLL REIS-Asia data set; Eviews 5 program, 2014.

The results are summarized in Table 2 and on the whole indicate that the variables of interest conform to a normal distribution, even though the JB test clearly indicates normality compliance at the 5% significance level but not at the 1% significance level (i.e., Normality 1); then at the 5% significance level but not at the 3% significance level (Normality 2); and lastly at the 1% significance level (Normality 3). As a result, the JLL REIS-Asia data set is found to exhibit partial normality behavior.

[1] JLL REIS-Asia is headquartered in Singapore and produces detail market research reports and forecasts of the prime real estate sector in key selected Asian cities.

Table 2 Summary of Normality Tests for Prime Real Estate Sectors, Singapore

	CV OFF	CV RES	CV RET	IY RES	IY RET	IY OFF	NR RES	NR RET	ER OFF	NR OFF
(Skewness/Std) ratio	√	√	√	√	√	√	√	√	√	√
(Kurtosis/Std) ratio	√	√	√	√	√	√	√	√	√	√
Jarque-Bera*	3	3	3	1	2	3	3	3	3	3

NB. √ denotes within the normality compliant range between –2 and +2.

* 1: accept normality at 5% significance level but reject normality at 1% significance level

2: accept normality at 5% significance level but reject normality at 3% significance level

3: accept normality at 1% significance level

CV = Capital Value ER = Effective Rent IY = Initial Yield

NR = Net Rent RES = Prime Luxury Residential Sector

OFF = Prime Office Sector RET = Prime Retail Sector

Source: Author; JLL REIS-Asia data set; Eviews 5 program, 2014.

3. The Beta (Systematic Risk) Model Estimations

3.1 The Modified Duration Model

First and foremost, the Duration (D_t) is used in the bond market to match asset liabilities. It measures the sensitivity of the value of an asset to changes in the interest rate. It is firstly developed by Macaulay (1938) and formulated as follows:

$$\frac{dV_t}{dy_t} \times \frac{1}{V_t} = \frac{-D_t}{(1+y_t)} \tag{1}$$

Where V_t: the value of the asset at time t; dy_t: the change in discount rate at time t.

The expression on the right hand side of Equation (1) is referred to as the modified duration D_t*. Rearranging the formula, the asset value's growth rate is obtained in terms of the modified duration as follows[2]:

$$\frac{dV_t}{V_t} = -D_t * dy_t \tag{2}$$

3.2 The Real Estate Return Volatility Relative to a Market Index

The anticipated rate of return of a real estate asset (property) *j* over a short period can be initially expressed as[3]:

$$R_{jt} = \frac{V_{jt} + a_{jt} + dV_{jt}}{V_{jt}} \tag{3}$$

Where a_{jt}: the initial income at time t; V_{jt}: the real estate asset value at time t;

dV_{jt}: the anticipated change in value at time t. Substituting from Equation (2) for the real estate asset j gives

$$R_{jt} = 1 + \frac{a_{jt}}{V_{jt}} - D_{jt} * dy_{jt} \tag{4}$$

For small values of t, a_{jt}, V_{jt} and D_{jt}* can be assumed to be constant. This implies that over time, the changes in the total return are influenced by the changes in the real estate yield. As real estate markets tend to be yield-driven, this assumption is not unreasonable. At time *t,* the variance of Equation (4) becomes

[2] The link between the bond price volatility and duration is firstly developed by Fisher (1966) and Hopewell and Kaufman (1973) later extended its discrete form.

[3] This anticipated rate of return is estimated similarly as the anticipated rate of return on a default-free bond over a short interval, for further details, see Livingston (1978).

$$Var(R_{jt}) = D_{jt}^{*2} Var(dy_{jt}) \tag{5}$$

A similar expression also exists for the variance of an index of real estate market movements R_{mt} such that

$$Var(R_{mt}) = (D_{mt}^{*})^2 Var(dy_{mt}) \tag{6}$$

The single index model suggests that the volatility of an investment relative to an index can be expressed as follows:

$$\beta_{jt} = \frac{\operatorname{cov}(R_{jt}, R_{mt})}{Var(R_{mt})} \tag{7}$$

This can be written as

$$\beta_{jt} = \frac{\rho(R_{jt}, R_{mt})\sigma(R_{jt})\sigma(R_{mt})}{Var(R_{mt})} \tag{8}$$

By substitution, the following expression is obtained:

$$\beta_{jt} = \frac{D_{mt} * D_{jt} * \rho(R_{jt}, R_{mt})\sigma(dy_{jt})\sigma(dy_{mt})}{(D_{mt}*)^2 Var(dy_{mt})} \tag{9}$$

Simplifying gives

$$\beta_{jt} = \frac{D_{jt}*}{D_{mt}*} \frac{\rho(dy_{jt}, dy_{mt})\sigma(dy_{jt})}{\sigma(dy_{mt})} \tag{10}$$

Equation (10) shows that the duration can play a theoretical role in determining the risk of a direct real estate asset investment and provides a rationale for non-stationarity of betas. According to Equation (10), the volatility of a direct real estate asset relative to a real estate market index is made up of two components. The first component is the modified duration of the property (i.e., the real estate asset) divided by a similar duration term for the real estate index (market duration). The second component is the covariance of changes in the equivalent yield of the direct real estate asset relative to the changes in the real estate market yield. This latter expression can also be interpreted to be the volatility of changes in the real estate yield. So, Equation (1.10) can be re-expressed as:

$$\beta_{jt} = \frac{D_{jt}*}{D_{mt}*} \beta_{dy_{jt} . dy_{mt}} \tag{11}$$

Note that Equation (11) provides an estimate of β_{jt} that is measured relative to a real estate market index. The justification for this approach is that real estate investors are frequently concerned about how well their portfolios perform relative to the real estate market. Via Equation (11), we can estimate the volatility of the real estate asset (or sector) return relative to the market that is useful in the performance measurement of the direct real estate portfolio. If the real estate index represents a reasonable proxy for the whole real estate market, and assuming equilibrium conditions, then there would be a linear relationship between the expected risk premium for both the real estate market and the market portfolio. This would imply that Equation (11) can be used to estimate the real estate systematic risk within a capital market framework. The advantage of Equation (11) in estimating the volatility of the real estate (or sector) return, relative to the market, is that it does not rely on a time series of historical data, and can be expressed in expectation form. As the duration is estimated from available data, the volatility of a real estate asset (or sector) can be readily estimated whenever a valuation is undertaken.

Estimation of the volatility of the real estate (or sector) return, relative to the market, via Equation (11), offers us some meaningful insights. Equation (11) reveals that the β of a direct real estate asset's return depends on the relative size of the duration of the direct real estate asset and the real estate market as well as the volatility of changes in the real estate yields. The importance of the latter implication is well observed in the valuation of an over rented real estate asset within the context of the United Kingdom (UK) practice. In this instance, a valuer may well argue that over an agreed time horizon, there would be changes in the market yield appropriate to the real estate asset so that the covariance between yield changes would be close to zero. As a result, β_j is also close to zero even though the respective durations take on positive values. The inference of this result is that in a capital market framework, the appropriate discount rate at which to value the real estate asset should be close to the risk free rate of return. In practice, we see over-rented properties being valued using the return on long-term government bond in 1990s in UK.

3.3 The Direct Real Estate Duration & Its Measurement

To use Equation (11), the estimation of the duration of a direct real estate asset is prerequisite. Based on Equation (2), the modified duration of the direct real estate asset *j* at time *t* can be formulated as:

$$-D_{jt}* = \frac{dV_{jt}}{dy_{jt}} \times \frac{1}{V_{jt}} \tag{12}$$

The direct real estate asset value, V_{jt}, can be estimated from the present value of the typical term and reversion freehold valuation model. The "typical term" is represented by an initial income stream, a_{jt}, that is fixed for n years at which time it is reviewed to the open market yield value, RV_{jt}. The present value is found by discounting at the equivalent yield, y_{jt}. Figure 1 depicts the equivalent yield model for a direct real estate asset in two parts. The first part consists of the current annual rental income a_{jt} for n years until the next rent review. The second part occurs at the next rental review when the annual rental income is replaced by the current estimate of rental value, RV_{jt}, which is then assumed to remain constant in perpetuity. For the direct real estate asset *j*, the present value at time *t*, V_{jt}, can be expressed as:

$$V_{jt} = a_j[\frac{1-(1+y_{jt})^{-n}}{y_{jt}}] + \frac{RV_{jt}}{y_{jt}(1+y_{jt})^n} \tag{13}$$

Rearranging gives

$$V_{jt} = \frac{a_{jt}}{y_{jt}} + \frac{(RV_{jt} - a_{jt})}{y_{jt}(1+y_{jt})^n} \tag{14}$$

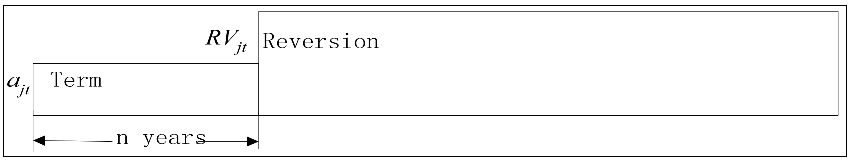

Figure 1 The Term and Reversion Parts of the Direct Real Estate Asset Equivalent Yield Model

Source: Authors, 2005 & 2014.

Equation (14) takes a non-linear form and is known as the real estate equivalent yield model, which is the most common method used for valuing the commercial real estate asset (i.e., property) and for analyzing current transactions. The equivalent yield in Equation (14) is usually lower than the risk adjusted return, reflecting the fact that there is growth in the income stream. In this model the equivalent yield as a discount rate for the expected cash flow incorporates the specific risk characteristics of the real estate asset, such as the lease term, rental growth, the physical condition and even the investor's expectation of the economy such as inflation expectation, forecasts of economy, and expected depreciation. While using the real estate equivalent yield model, it is the UK practice and throughout many of the British Commonwealth countries, including Singapore, to set RV_{jt} equal to the current rental value even though it arises n periods in the future. The equivalent yield incorporates readily available information that is expressed in current day terms. In a market that is yield driven, it may well be reasonable to assume that most valuers are familiar with equivalent yields, and the equivalent yields embody more than adequate information with respect to the lease structure of individual real estate assets, together with the expectations of rental value growth and expected returns.

It should be firstly noticed that although Equation (14) can be shown to be mis-specified[4] in economic terms, there is no guarantee that it would produce valuations that differ from a model that explicitly allows for growth in rental values. The choice of the yield in these models is vital. Because of the importance of the direct real estate equivalent yield, valuers are interested to know by how much a small change in the yield can affect capital value. It is thus appropriate to examine the duration of a direct real estate asset relative to changes in the equivalent yield. From Equation (14), the first derivative of V_{jt} with respect to y_{jt} can be expressed as:

$$\frac{dV_{jt}}{dy_{jt}} = -\frac{a_{jt}}{y^2_{jt}} - \frac{(RV_{jt} - a_{jt})}{y_{jt}(1+y_{jt})^n}\left[\frac{1}{y_{jt}} + \frac{n}{(1+y_{jt})}\right] \tag{15}$$

Dividing through by the real estate asset value V_{jt} and substituting $1/V_{jt}$ by $\dfrac{y_{jt}(1+y_{jt})^n}{a_{jt}(1+y_{jt})^n + (RV_{jt} - a_{jt})}$ would give the modified duration as:

$$D^*_{jt} = \left\{\frac{a_{jt}}{y^2_{jt}} + \frac{(RV_{jt} - a_{jt})}{y_{jt}(1+y_{jt})^n}\left[\frac{1}{y_{jt}} + \frac{n}{(1+y_{jt})}\right]\right\} \bullet \frac{y_{jt}(1+y_{jt})^n}{a_{jt}(1+y_{jt})^n + (RV_{jt} - a_{jt})} \tag{16}$$

Noting that for a fully rack-rented[5] real estate asset in which the rental value, RV_{jt}, is equal to the passing income, a_{jt}, the modified duration can be reduced to

$$D^*_{jt} = \frac{1}{y_{jt}} \qquad (17)^6$$

3.4 The Real Estate Sector Duration Beta

Even though the model, as defined by Equation (11) and Equation (16), is augmented to incorporate constant expectations with respect to the risk premium and the risk free rate of return, some key trends concerning the prime office, luxury residential and retail sectors are discussed below. The luxury residential sector has shown a sharp decline in the return volatility (i.e., the duration beta β_{jt}) for the period between 1991 and 1993, and to be followed by a steady rise until 1995. If investors maintain constant expectations concerning the risk premium and the risk free rate of return, the change in the return volatility would imply that the expected value of the luxury

[4] Misspecification arises when we set RV_{jt} equal to the current rental value for economic inconsistencies. However, economic deficiencies in the model, as well as differences in the lease structure, are accommodated in the choice of equivalent yield. There are widely publicized equivalent yields with property transactions and at the index level, time series of equivalent yields are also readily available and form an important part of published information for real estate, for this reason, the equivalent yield model is the most common approach used to value property.

[5] Rack-rent is originally a rent which a property would command in a free market. It is the highest amount that can be paid for land from labor's production that will enable him to survive (and reproduce). Even as new skills and techniques are adopted, and innovative technology is put to work, so will rack-rent rise, swallowing the lion's share of the product. For a fully rack-rented property, where the passing rent equals the rental value, valuers would value the income stream until the review date as an annuity, and would capitalize the increase in rent in perpetuity at the equivalent yield.

[6] For mathematic proof of Equation (17), please refer to the Appendix.

residential sector did peak in 1993 and has since proceeded to decline. In contrast, the return volatility of the prime retail sector and that of the prime office sector have shown a sharp rise in the period between 1991 and 1993, followed by an overall decline through to 1997 and 1999. In general, the return volatility trends imply that the expected value of these two prime sectors did bottom in the period between 1992 and 1993. As for prime office sector, the trend is steadily downward, with the inference that the expected value of prime office real estate assets has been firming up. However, an opposite trend in general is observed for the prime retail sector, with the inference that the expected value for prime retail real estate assets has been weakening.

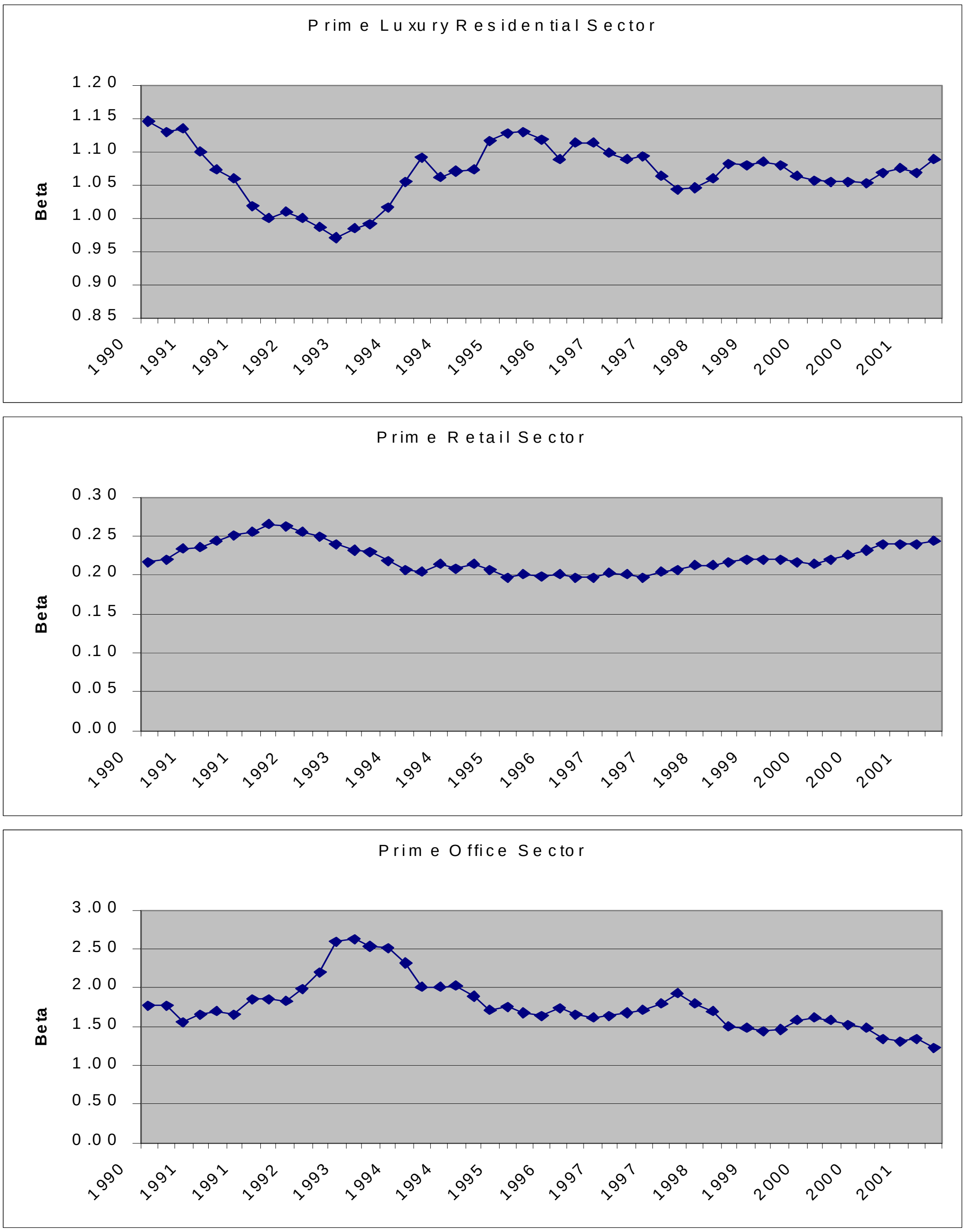

Figure 2 Beta Estimates of the Real Estate Sectors' Return Volatility
Source: Author; JLL REIS-Asia data set, 2006 & 2014

3.5 The Time-varying Beta Regression

As the JLL REIS-Asia data set is partially normal distributed and therefore time variant, it may well be appropriate to investigate the time-varying nature of the return volatility of a real estate asset (sector) relative to the market subsequently. The next stage would be to investigate the validity of the duration-based, beta-volatility results by comparing them with the time-varying betas that in turn are obtained through the ordinary least-square regression analysis, which is estimated for each real estate sector's returns versus the real estate market's returns. The purpose would be to test whether the real estate duration model picks up general trends in volatility, through comparing the two different betas. If both models pick up the same informational trends, then it would be expected to show some similarity in the general trend.

The time-varying regression betas are estimated from the following expression:

$$r_t = \alpha_t + \beta_t r_{mt} + \omega_t \tag{18}$$

Where ω_t is a random error term. The coefficients in Equation (18) have a time subscript implying that they can vary over time. If it is assumed that information arrives randomly, then the evolution of both parameters would follow a random walk. The coefficients for both α_t and β_t can be expressed as:

$$\alpha_t = \alpha_{t-1} + \lambda_t \ \ where \ \ \lambda \sim NID(0, \sigma_\lambda{}^2) \tag{19}$$

$$\beta_t = \beta_{t-1} + \varepsilon_t \ \ where \ \ \varepsilon \sim NID(0, \sigma_\varepsilon{}^2) \tag{20}$$

Where λ_t and ε_t are random error terms that are normal and identically distributed with $\varepsilon(\lambda_t) = \varepsilon(\varepsilon_t) = \varepsilon(\lambda_t, \varepsilon_t) = 0$. The intercept and slope coefficients are able to pick up changes in market conditions. By reformulating the system in state-space form, Equation (18) can be expressed as a measurement equation and with Equation (19) and Equation (20) as transition equations (Harvey, 1993). However, this paper's main interest in this case is with the slope coefficient β_t. It is a time varying parameter that measures the return volatility of each real estate sector of the wider real estate market at each point of time *t*, on the basis of a historical series of returns. It is also compared with the duration estimate, β_{jt}, of Equation (11) that is based on the expected cash flows for each prime real estate sector. Both estimates of volatility give single point figures but because it is a comparison of historic and expected values, it is certain that both estimates of volatility would not match on a quarter-by-quarter basis. Nevertheless, both the resulting profiles should follow the same general trend.

The results of the comparison between the two volatility approaches for each prime real estate sector are depicted in Figure 3. The corresponding details of these results are provided in Appendix 3 for reference purposes. Except for the prime office sector, both the duration beta and the time-varying beta profiles follow the same general trend. In general, the luxury residential sector and the prime office sector are inclined to move in opposite direction in terms of both the two different beta measurements, which has a significant meaning in real estate asset allocation[7].

Nevertheless, the prime office sector shows greater volatility in the duration beta compared with the time-varying beta. This may imply that investors are expecting greater volatility in expected returns than is realized in the historic returns. However, the two beta measurements take an opposite trend for the prime retail sector. Empirically, the time-varying beta of this retail sector shows greater volatility than the associated duration beta and tends to be overstated compared with the duration beta. There are two possible reasons:

[7] Interestingly, based on the 1972-78 USequity REITs returns, Miles and McCue (1982) reports a negative correlation Coefficient between office and residential property sectors.

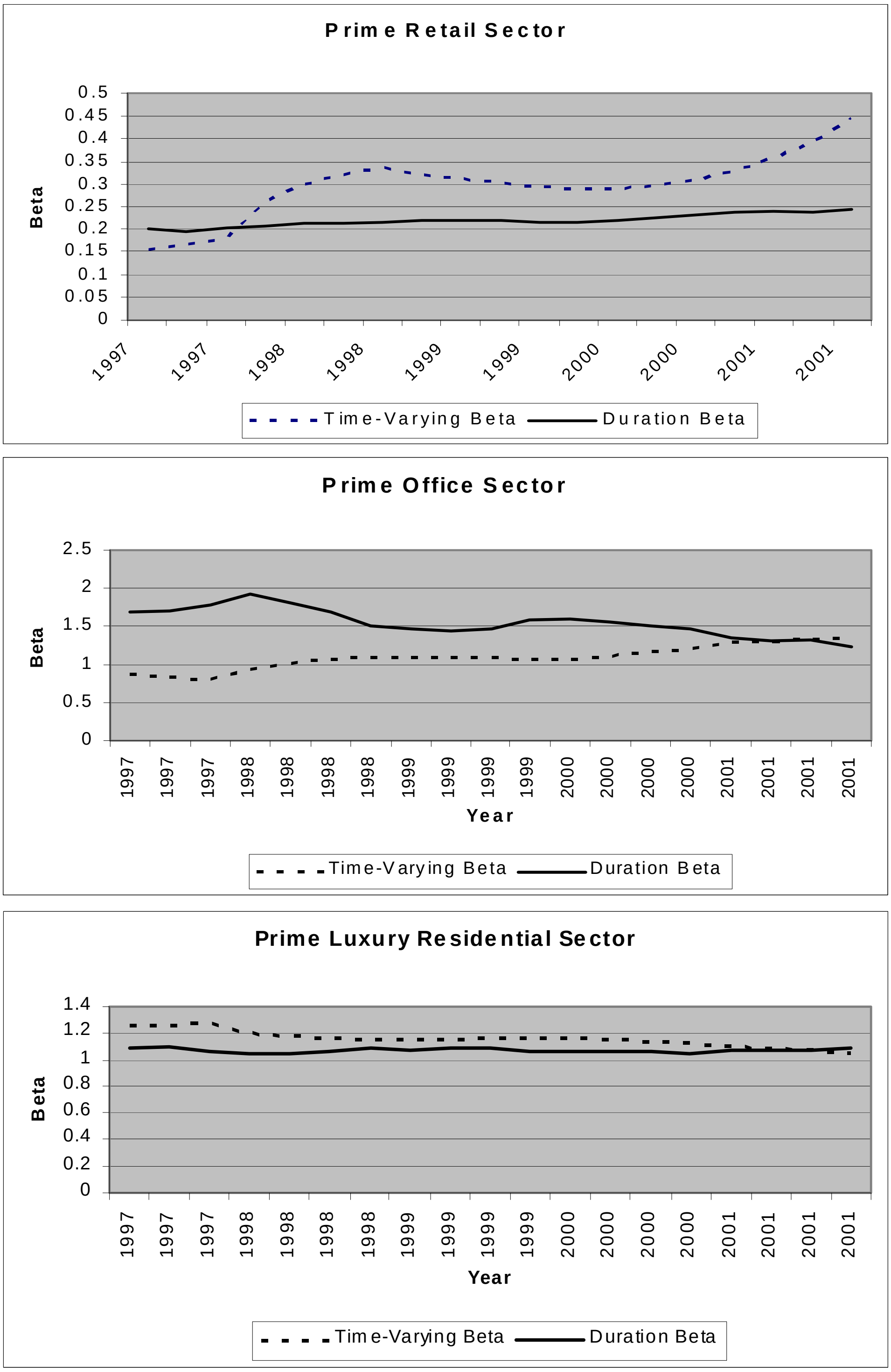

Figure 3 Comparison of Duration Beta and Time-Varying Regression Beta

Source: Author; JLL REIS-Asia data set, 2006 & 2014

- First, the time-varying regression beta is a generalized model, the coefficient will contain the impact of many other factors and therefore fails to accurately estimate the systematic risk, i.e., the risk that originates from the wider real estate market risk;

- secondly, inaccuracies may well come from the evolving α and β, in Equations (18) to (20), that may not have been properly represented by the time-varying regression model.

Table 3 Correlation Matrix of Duration Beta and Time-varying Regression Beta

	Duration Beta			Time Varying Beta		
	Dur-Ret	Dur-Off	Dur-Res	TV-Ret	TV-Off	TV-Res
Dur-Ret	1.00	-0.87	0.00	0.85	0.97	-0.96
Dur-Off		1.00	-0.45	-0.74	-0.87	0.84
Dur-Res			1.00	-0.03	0.00	0.01
TV-Ret				1.00	0.910	-0.95
TV-Off					1.00	-0.99
TV-Res						1.00

Source: Author, 2014

The implications from the foregoing analysis are further investigated by inspecting the correlation statistics in Table 3. As for the prime retail sector, the correlation between the duration beta and the time-varying beta is strongly positive with a value of 0.85. On the contrary, the duration beta and the associated time varying beta for the prime office sector are negatively correlated at -0.87. Table 3 also highlights a strong negative correlation of -0.87 between the duration beta estimates for the prime retail sector and the prime office sector. The expected return for these two sectors should, therefore, be negatively correlated. This finding has important diversification implications for a long-term investment in these two sectors.

Table 4 Comparison of Duration Beta and Time-varying Regression Beta

Sector	Average Duration Beta	Std. Dev.	Average Time-varying Regression Beta	Std. Dev.
Prime Retail	0.220	0.014	0.297	0.071
Prime Luxury Residential	1.068	0.015	1.157	0.061
Prime Office	1.550	0.183	1.097	0.286

Source: Author, 2014

Table 4 shows that in the period between April 1997 and October 2001, the ranking of betas is different under each volatility approach. The ranking of the duration betas follows the order of "office-residential-retail", while the ranking of time-varying regression betas follow a different order of "residential-office-retail". However, these rankings may well be just a difference of the rank-order between the *ex ante* and *ex post* approaches of the volatility analysis.

4. The Real Estate Asset Total Risk Estimation under the Duration and GARCH Models

4.1 The Direct Real Estate Asset Total Risk Duration Model

In a duration model, a linear relationship is presumed between changes in both the fixed-income asset value and the market-wide interest rate. For large changes in the interest rate, the model does not accurately reflect changes in value, and such changes can be reflected through the convexity concept. However, by writing the change in the capital value of the direct real estate asset j as the first two terms of a Taylor expansion, then the following expression can be derived:

$$dV_{jt} = \frac{dV_{jt}}{dy_{jt}} dy_{jt} + \frac{1}{2}\frac{d^2V_{jt}}{dy^2{}_{jt}}(dy_{jt})^2 \tag{21}$$

Dividing through by V_{jt} and substituting D_{jt}* for the modified duration and C_{jt} for convexity, would produce the expression:

$$\frac{dV_{jt}}{V_{jt}} = -D^{*}{}_{jt} dy_{jt} + \frac{1}{2} C_{jt} (dy_{jt})^2 \tag{22}$$

Where $C_{jt} = \frac{d^2 V_{jt}}{dy^2{}_{jt}} \frac{1}{V_{jt}}$

Assuming a fully rack rented direct real estate asset (property), the percentage change in capital value is

$$\frac{dV_{jt}}{V_{jt}} = -\frac{1}{y_{jt}} dy_{jt} + \frac{1}{y_{jt}^2} (dy_{jt})^2 \qquad (23)^{8}$$

Taking convexity into consideration may improve our calculations and knowing the distribution of the direct real estate yield changes, it would be possible to simulate a distribution for the percentage change in value. However, the main concern in looking at convexity is in the effect that it could have on the direct real estate total risk. This is imperative for large changes in the direct real estate yield. However, the average change in the yield for the Singapore real estate market is only -0.21% per quarter. With such a small value, the effect of convexity only influences the third decimal place in the growth calculations. As long as the direct real estate yield changes are relatively small, then it is likely that convexity would not have a great influence on the estimate of direct real estate total risk, and can thus be ignored. To provide an estimate of the total risk, a further assumption is that the direct real estate asset is fully rented so that the current income is equal to the rental value. Given these simplifications for practical purposes, the direct real estate total risk is expressed as:

$$Var(g_{jt}) = (D_{jt}{*})^2 Var(dy_{jt}) \tag{24}$$

Where $Var(g_{jt})$ is the variance of the capital value growth. Equation (24) shows that the volatility of the direct real estate yields is an important component in explaining the direct real estate total risk changes. If changes in the direct real estate yields were always close to zero, then Equation (24) implies that changes in real estate capital values would have scarcely any volatility.

In order to estimate the real estate asset, sector or market total risk, this paper utilizes "The Real Estate Asset Total Risk Duration Model" of Equation (3), which is derived under section 1.3 The Theoretical Framework of Analysis for this research & Research Hypothesis. The JLL REIS-Asia dataset that is discussed in section 3.2 on "The Data Set" is deployed for this purpose. From the JLL REIS-Asia quarterly data set, it is found that the average duration of the Singapore real estate market is close to 21.7 years. The variance of the change in the Singapore real estate market yields is found to be around 0.0013. Substituting these two estimated results into Equation (24), and taking the square root, would give an average standard deviation of 0.782% quarterly.

4.2 Duration Risk and GARCH Risk

Equation (24) estimates the variance of the quarterly capital value growth of the wider Singapore real estate market at a single point in time, *t*. The model can also develop time varying estimates of total risk. To investigate how good these estimates are, they need to be compared with an alternative method of estimating the total risk over time. The alternative estimation method is the GARCH model, which can be utilized to estimate the conditional variance of the quarterly capital value growth of Singapore's real estate market (see Bollerslev &

[8] Please refer to the Appendix for details on the derivation of this formula.

Wooldridge, 1992). The total risk of the Singapore real estate market is estimated under both the "Duration Risk" model (i.e., the real estate asset total risk duration model) and the "GARCH Risk" model, and they are depicted in Figure 4. The profile developed using Equation (24) can be considered to be an estimate of the expected total risk whereas the GARCH risk model measures the realized total risk. More specifically, the GARCH (1, 1) model is utilized and defined in Equation (25).

$$\sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{25}$$

Since σ_t^2 is the one-period (quarter) ahead forecast-variance, based on past information, it is known as the conditional variance function comprising three terms: a constant term *w*; the ARCH term ε_{t-1}^2 where news about the volatility from the previous period are measured as the lag of the squared residual from a mean equation; and the GARCH term σ_{t-1}^2 representing the last period's forecast variance. The "(1, 1)" in the GARCH (1, 1) denotes the presence of a first-order autoregressive GARCH term and a first-order moving average ARCH term. This model's errors also follow a heteroskedastic ARMA (1, 1) process where any volatility shock should die out slowly. The GARCH (1, 1) model estimates are presented in Table 5 below.

Table 5 Estimated GARCH (1, 1) Model for Total Risk, Singapore's Real Estate Market

Dependent Variable: Total Returns, Singapore Real Estate Market (% per qtr)
Method: ML-ARCH
Sample: 1990:3 2001:4
Included observations: 46
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	17.49880	88.26730	0.198248	0.8429
	Variance Equation			
C	12723.21	15080.04	0.843712	0.3988
ARCH(1)	-0.034576	0.057473	-0.601609	0.5474
GARCH(1)	0.589448	0.529965	1.112239	0.2660
R-squared	-0.000473	Mean dependent var		20.54109
Adjusted R-squared	-0.071935	S.D. dependent var		141.4573
S.E. of regression	146.4568	Akaike info criterion		12.82804
Sum squared resid	900882.9	Schwarz criterion		12.98705
Log likelihood	-291.0448	Durbin-Watson stat		2.044511

Source: Author, Eviews 5 Program, 2014.

Of relevance would be the Akaike and Schwarz information criteria in Table 5 that test for model selection on the basis of striking a balance between goodness of fit and parsimony. These two test criteria are not excessive (as low values are preferred), implying that the GARCH (1, 1) model is appropriately selected. It can be readily observed from Figure 4 that the period after 1995 shows a strong positive correlation between the "Duration Risk" model and the "GARCH Risk" model. However, prior to 1995, the correlation is weak. The difference may well be attributed to the nature of the risk measurement itself because the two models measure what investors expect and what is realized respectively. Since May 1996, the introduction of the Singapore government's anti-speculation policy to deter speculation in the real estate market, has contributed a significant part to a declining trend in the duration measure of the real estate market total risk for Singapore.

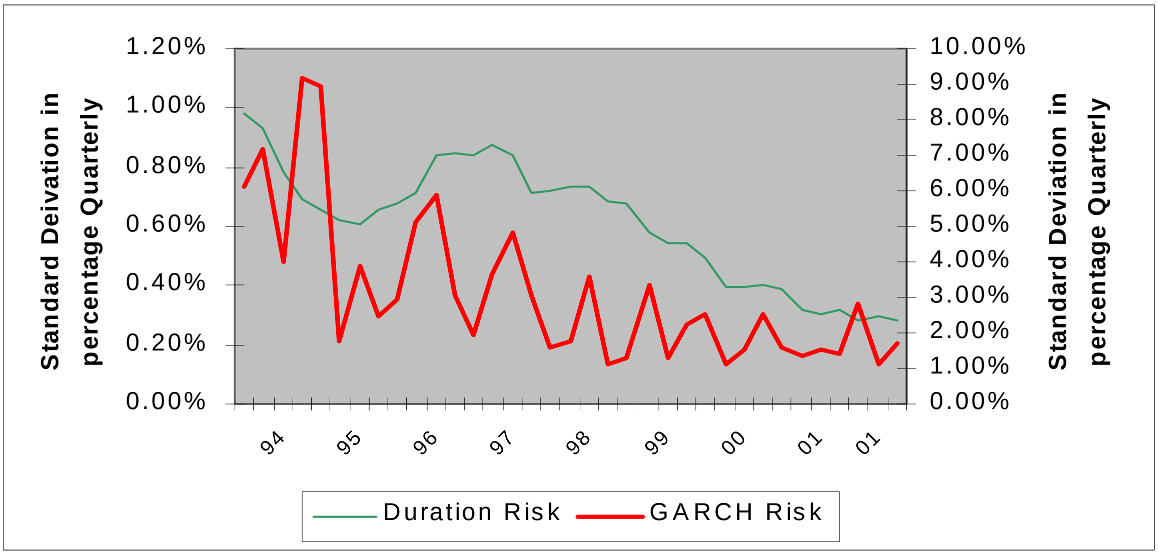

Figure 4 Comparison of GARCH and Duration Measures of Total Risk

Source: Author; Eviews 5 program, 2014

5. Conclusion

The modified duration, which is often used in the price analyses of fixed-income assets (i.e., bonds) and stocks, has the potential of being uniquely modified to be the direct real estate duration model for a real estate sector or its wider real estate market. The direct real estate duration model can then be structurally modeled and utilized to estimate the return volatility of a direct real estate asset (or sector) relative to the real estate market, i.e., the direct real estate sector's systematic risk as well as the particular real estate sector's or market's total risk. The direct real estate duration model can even be based on information readily available and known to the valuer. No past time series data is involved. The limitation is that the accuracy of the approach would depend on a valuer's ability to anticipate changes in the real estate asset (or sector) yield. Another limitation is that the model incorporates constant expectations with respect to real estate risk premiums and the risk free rate of return. Hence, the direct real estate duration model offers good potential in a number of areas such as estimating the direct real estate expected returns, asset allocation, risk monitoring and performance measurement.

From an in-depth investigation of the prime real estate sectors of Singapore, and utilizing the JLL REIS-Asia data set, the derived duration betas for the prime office sector and the prime retail sector are on the whole very stable, relative to the prime luxury residential sector. Furthermore, the negative correlation analysis between the duration beta of the prime retail and office sectors highlights the importance of diversification for a long term investment in these two prime sectors.

As for the direct real estate systematic risk, this paper compares the direct real estate duration beta estimates with the time-varying beta regression estimates for each of the three prime real estate sectors. Except for the prime office sector, both the duration beta and the time-varying beta profiles follow the same general trend. In general, the luxury residential sector and the prime office sector are inclined to move in opposite direction. However, the prime office sector shows greater volatility in the duration beta compared with the time-varying beta. This may imply that investors are expecting greater volatility in expected returns than is realized in the historic returns.

Nevertheless, it is the just the opposite trend for the prime retail sector, where the time-varying beta of this sector shows greater volatility than its associated duration beta. Another key observation of this retail sector is that the time-varying beta tends to be overstated when compared with the duration beta. There are two possible reasons: first, the time-varying regression beta is a generalized model, and therefore fails to accurately estimate the systematic risk, i.e., the wider direct real estate market risk; secondly, inaccuracies may well come from the evolving α and β, in Equation (25), which may not have been properly represented by the time-varying regression model.

Finally, the total risk of the wider direct Singapore real estate market is estimated under both the "Real Estate Asset Total Risk Duration Model" and the "GARCH Risk" model. It is readily observed that the period after 1995 shows a strong positive correlation between the "Real Estate Asset Total Risk Duration Model" model and the "GARCH Risk" model. However, prior to 1995, the correlation is weak. The difference may well be attributed to the nature of the risk measurement itself because the two models measure what investors expect and what is realized respectively. Since May 1996, the introduction of the Singapore government's anti-speculation policy to deter speculation in the real estate market, has contributed a significant part to a declining trend in the duration measure of the real estate market total risk for Singapore.

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Appendix

The Mathematical Proof for Equation (17)

Based upon the equivalent yield model for a direct real estate asset *j*

$$V_{jt} = \frac{a_{jt}}{y_{jt}} + \frac{(RV_{jt} - a_{jt})}{y_{jt}(1 + y_{jt})^n} \tag{14}$$

the first derivation of the capital value, V_{jt}, with respect to y_{jt} can be defined in Equation (15) as

$$\frac{dV_{jt}}{dy_{jt}} = -\frac{a_{jt}}{y^2{}_{jt}} - \frac{(RV_{jt} - a_{jt})}{y_{jt}(1 + y_{jt})^n}\left[\frac{1}{y_{jt}} - \frac{n}{(1 + y_{jt})}\right] \tag{15}$$

Further, rearranging Equation (14) produces

$$V_{jt} = \frac{a_{jt}}{y_{jt}} + \frac{(RV_{jt} - a_{jt})}{y_{jt}(1 + y_{jt})^n} = \frac{a_{jt}(1 + y_{jt})^n + (RV_{jt} - a_{jt})}{y_{jt}(1 + y_{jt})^n}$$

hence,

$$\frac{1}{V_{jt}} = \frac{y_{jt}(1 + y_{jt})^n}{a_{jt}(1 + y_{jt})^n + (RV_{jt} - a_{jt})}$$

In combination with Equation (12),

$$-D_{jt}* = \frac{dV_{jt}}{dy_{jt}} \times \frac{1}{V_{jt}} \tag{12}$$

Multiplying $\frac{1}{V_{jt}}$ on the both sides of Equation (15) will produce the modified duration, $D_{jt}{}^*$, for the direct real estate asset *j*, as shown in Equation (16).

$$D^*{}_{jt} = \left\{\frac{a_{jt}}{y^2{}_{jt}} + \frac{(RV_{jt} - a_{jt})}{y_{jt}(1 + y_{jt})^n}\left[\frac{1}{y_{jt}} - \frac{n}{(1 + y_{jt})}\right]\right\} \bullet \frac{y_{jt}(1 + y_{jt})^n}{a_{jt}(1 + y_{jt})^n + (RV_{jt} - a_{jt})} \tag{16}$$

In the instance of a fully rack-rented real estate asset where the direct real estate rental value, RV_{jt}, is equal to the passing annual rental income, a_{jt}, then the modified duration, D^*_{jt}, can be further simplified.

$$D^*{}_{jt} = \left\{\frac{a_{jt}}{y^2{}_{jt}}\right\} \bullet \frac{y_{jt}(1 + y_{jt})^n}{a_{jt}(1 + y_{jt})^n} = \frac{1}{y_{jt}} \tag{17}$$

Mathematical Derivation of Equation (23)

By writing the change in value of the direct real estate asset *j* as the first two terms of a Taylor expansion gives the expression:

$$dV_{jt} = \frac{dV_{jt}}{dy_{jt}} dy_{jt} + \frac{1}{2}\frac{d^2V_{jt}}{dy^2{}_{jt}}(dy_{jt})^2. \tag{22.1}$$

Dividing through by V_{jt} and substituting $D_{jt}*$ for the modified duration and C_{jt} for convexity produces the expression:

$$\frac{dV_{jt}}{V_{jt}} = -D^*{}_{jt} dy_{jt} + \frac{1}{2}C_{jt}(dy_{jt})^2. \tag{22.2}$$

Where $C_{jt} = \frac{d^2 V_{jt}}{dy^2{}_{jt}} \frac{1}{V_{jt}}$.

The second term on the right hand side of Equation (22.2) can be reduced into:

$$\frac{1}{2} C_{jt} (dy_{jt})^2 = \frac{1}{2} \frac{d^2 V_{jt}}{dy_{jt}{}^2} \frac{1}{V_{jt}} (dy_{jt})^2$$

$$= \frac{1}{2} \frac{d}{dy_{jt}} (\frac{dV_{jt}}{dy_{jt}}) \cdot \frac{1}{V_{jt}} (dy_{jt})^2$$

$$= \frac{1}{2} \frac{1}{V_{jt}} \cdot \frac{d}{dy_{jt}} (\frac{dV_{jt}}{dy_{jt}} \cdot \frac{1}{V_{jt}} \cdot V_{jt})(dy_{jt})^2$$

Based on Equations (12) and (17) and under assumption of a fully rack rented property, further simplification can be obtained:

$$\frac{1}{2} C_{jt} (dy_{jt})^2 = \frac{1}{2} \frac{1}{V_{jt}} \cdot \frac{d}{dy_{jt}} (-D^*_{jt} \cdot V_{jt})(dy_{jt})^2$$

$$= \frac{1}{2} \frac{1}{V_{jt}} \cdot \frac{d}{dy_{jt}} (-\frac{1}{y_{jt}} \cdot V_{jt})(dy_{jt})^2$$

$$= -\frac{1}{2} \frac{1}{V_{jt}} \cdot (-\frac{V_{jt}}{y_{jt}^2} + \frac{1}{y_{jt}} \cdot \frac{dV_{jt}}{dy_{jt}} \cdot \frac{1}{V_{jt}} \cdot V_{jt})(dy_{jt})^2$$

$$= -\frac{1}{2} \frac{1}{V_{jt}} \cdot [-\frac{V_{jt}}{y_{jt}^2} + \frac{1}{y_{jt}} \cdot (-D^*_{jt}) \cdot V_{jt}](dy_{jt})^2$$

$$= -\frac{1}{2} \frac{1}{V_{jt}} \cdot [-\frac{V_{jt}}{y_{jt}^2} + \frac{1}{y_{jt}} \cdot (-\frac{1}{y_{jt}}) \cdot V_{jt}](dy_{jt})^2$$

$$= \frac{1}{y_{jt}^2} (dy_{jt})^2$$

Rearranging and substituting $D_{jt}{}^*$ with $\frac{1}{y_{jt}}$, we transform Equation (22.2) into:

$$\frac{dV_{jt}}{V_{jt}} = -\frac{1}{y_{jt}} dy_{jt} + \frac{1}{y_{jt}^2} (dy_{jt})^2 \tag{23}$$

The Duration Betas of the Three Prime Real Estate Sectors

			Modified Duration(D*)	Market Modified Duration	**Prime Luxury Residential** BETA
APR	**1990**	0.55	23.26	15.6	0.81
JUL	1990	0.55	21.28	14.7	0.79
OCT	1990	0.55	19.23	13.1	0.80
JAN	**1991**	0.55	17.54	12.7	0.76
APR	1991	0.55	16.95	12.9	0.72
JUL	1991	0.55	16.67	13.1	0.69
OCT	1991	0.55	16.13	13.9	0.63
JAN	**1992**	0.55	15.63	13.9	0.61
APR	1992	0.55	16.13	14.1	0.62
JUL	1992	0.55	16.67	14.9	0.61
OCT	1992	0.55	16.95	15.8	0.59
JAN	**1993**	0.55	17.54	16.9	0.57
APR	1993	0.55	18.87	17.6	0.59
JUL	1993	0.55	19.61	17.9	0.60
OCT	1993	0.55	21.74	18.6	0.64
JAN	**1994**	0.55	24.39	19.0	0.70
APR	1994	0.55	25.64	18.4	0.76
JUL	1994	0.55	25.64	19.7	0.71
OCT	1994	0.55	26.32	19.7	0.73
JAN	**1995**	0.55	25.64	19.0	0.74
APR	1995	0.55	27.78	18.7	0.81
JUL	1995	0.55	29.41	19.3	0.83
OCT	1995	0.55	28.57	18.6	0.84
JAN	**1996**	0.55	28.57	18.8	0.83
APR	1996	0.55	28.57	20.0	0.78
JUL	1996	0.55	30.30	20.1	0.82
OCT	1996	0.55	29.41	19.5	0.82
JAN	**1997**	0.55	28.57	19.6	0.80
APR	1997	0.55	28.57	19.9	0.79
JUL	1997	0.55	29.41	20.2	0.80
OCT	1997	0.55	28.57	21.0	0.74
JAN	**1998**	0.55	27.78	21.5	0.71
APR	1998	0.55	27.03	20.6	0.71
JUL	1998	0.55	27.03	20.0	0.74
OCT	1998	0.55	27.03	18.8	0.78
JAN	**1999**	0.55	26.32	18.4	0.78
APR	1999	0.55	26.32	18.2	0.79
JUL	1999	0.55	26.32	18.3	0.78
OCT	1999	0.55	26.32	19.1	0.75
JAN	**2000**	0.55	26.32	19.3	0.74
APR	2000	0.55	25.97	19.2	0.74
JUL	2000	0.55	25.64	18.9	0.74
OCT	2000	0.55	25.32	18.8	0.74
JAN	**2001**	0.55	25.00	17.8	0.77
APR	2001	0.55	25.00	17.5	0.78
JUL	2001	0.55	25.00	17.7	0.77
OCT	2001	0.55	25.00	16.8	0.81

		BETA ($\beta\delta_{yjt},\delta_{ymt}$)	Modified Duration(D*)	Market Modified Duration	**PRIME Retail** β_{jt}(Beta)
APR	**1990**	0.21	11.90	15.60	0.16
JUL	1990	0.21	11.24	14.68	0.16
OCT	1990	0.21	10.75	13.05	0.17
JAN	**1991**	0.21	10.20	12.65	0.17
APR	1991	0.21	10.42	12.91	0.17
JUL	1991	0.21	10.75	13.13	0.17
OCT	1991	0.21	10.99	13.89	0.16
JAN	**1992**	0.21	11.24	13.93	0.17
APR	1992	0.21	11.36	14.10	0.17
JUL	1992	0.21	11.49	14.85	0.16
OCT	1992	0.21	11.63	15.77	0.15
JAN	**1993**	0.21	11.76	16.88	0.15
APR	1993	0.21	12.05	17.55	0.14
JUL	1993	0.21	12.35	17.91	0.14
OCT	1993	0.21	12.66	18.62	0.14
JAN	**1994**	0.21	12.99	19.04	0.14
APR	1994	0.21	12.99	18.40	0.15
JUL	1994	0.21	14.08	19.66	0.15
OCT	1994	0.21	13.89	19.66	0.15
JAN	**1995**	0.21	13.89	18.95	0.15
APR	1995	0.21	13.89	18.72	0.15
JUL	1995	0.21	13.89	19.30	0.15
OCT	1995	0.21	13.70	18.61	0.15
JAN	**1996**	0.21	13.70	18.84	0.15
APR	1996	0.21	14.29	19.99	0.15
JUL	1996	0.21	14.49	20.11	0.15
OCT	1996	0.21	14.08	19.48	0.15
JAN	**1997**	0.21	14.29	19.59	0.15
APR	1997	0.21	14.29	19.85	0.15
JUL	1997	0.21	14.29	20.19	0.15
OCT	1997	0.21	14.93	20.98	0.15
JAN	**1998**	0.21	14.93	21.47	0.14
APR	1998	0.21	14.81	20.64	0.15
JUL	1998	0.21	14.71	19.95	0.15
OCT	1998	0.21	14.60	18.80	0.16
JAN	**1999**	0.21	14.49	18.37	0.16
APR	1999	0.21	14.49	18.16	0.17
JUL	1999	0.21	14.49	18.31	0.16
OCT	1999	0.21	14.49	19.05	0.16
JAN	**2000**	0.21	14.49	19.30	0.16
APR	2000	0.21	14.71	19.18	0.16
JUL	2000	0.21	14.93	18.93	0.16
OCT	2000	0.21	15.15	18.78	0.17
JAN	**2001**	0.21	15.15	17.80	0.18
APR	2001	0.21	15.15	17.50	0.18
JUL	2001	0.21	15.15	17.74	0.18
OCT	2001	0.21	15.15	16.85	0.19

		BETA (βδyjt,δymt)	D_{jt} Modified Duration	D_{mt} Market Modified Duration	**PRIME OFFICE** β_{jt}(Beta) **office**
APR	**1990**	1.46	22.95	15.60	2.14
JUL	1990	1.46	21.39	14.68	2.12
OCT	1990	1.46	16.83	13.05	1.88
JAN	**1991**	1.46	16.91	12.65	1.95
APR	1991	1.46	17.20	12.91	1.94
JUL	1991	1.46	16.67	13.13	1.85
OCT	1991	1.46	18.75	13.89	1.97
JAN	**1992**	1.46	18.53	13.93	1.94
APR	1992	1.46	18.72	14.10	1.94
JUL	1992	1.46	21.15	14.85	2.07
OCT	1992	1.46	24.18	15.77	2.23
JAN	**1993**	1.46	29.97	16.88	2.59
APR	1993	1.46	32.15	17.55	2.67
JUL	1993	1.46	32.15	17.91	2.62
OCT	1993	1.46	34.29	18.62	2.68
JAN	**1994**	1.46	34.29	19.04	2.63
APR	1994	1.46	30.26	18.40	2.40
JUL	1994	1.46	31.16	19.66	2.31
OCT	1994	1.46	31.89	19.66	2.36
JAN	**1995**	1.46	28.87	18.95	2.22
APR	1995	1.46	27.26	18.72	2.12
JUL	1995	1.46	29.29	19.30	2.21
OCT	1995	1.46	27.08	18.61	2.12
JAN	**1996**	1.46	26.62	18.84	2.06
APR	1996	1.46	28.94	19.99	2.11
JUL	1996	1.46	28.94	20.11	2.10
OCT	1996	1.46	27.39	19.48	2.05
JAN	**1997**	1.46	27.39	19.59	2.04
APR	1997	1.46	28.13	19.85	2.07
JUL	1997	1.46	29.43	20.19	2.12
OCT	1997	1.46	30.81	20.98	2.14
JAN	**1998**	1.46	32.77	21.47	2.22
APR	1998	1.46	29.68	20.64	2.10
JUL	1998	1.46	27.61	19.95	2.02
OCT	1998	1.46	24.08	18.80	1.87
JAN	**1999**	1.46	23.03	18.37	1.83
APR	1999	1.46	22.43	18.16	1.80
JUL	1999	1.46	22.84	18.31	1.82
OCT	1999	1.46	25.01	19.05	1.91
JAN	**2000**	1.46	25.69	19.30	1.94
APR	2000	1.46	24.72	19.18	1.88
JUL	2000	1.46	23.49	18.93	1.81
OCT	2000	1.46	22.68	18.78	1.76
JAN	**2001**	1.46	20.20	17.80	1.65
APR	2001	1.46	19.48	17.50	1.62
JUL	2001	1.46	20.02	17.74	1.65
OCT	2001	1.46	18.06	16.85	1.56

The Time Varying Betas of the Three Prime Real Estate Sectors

Office (SqM)	Prime Retail	Prime Luxury Residential	Prime Office	Singapore Real Estate Market	Prime Retail	Time-Varying Beta	Prime Office
0.18	2.80%	-2.80%	-4.20%	-1.90%			
0.18	3.10%	-2.50%	-3.90%	-1.60%			
0.18	3.40%	-2.10%	-3.10%	-1.20%			
0.18	9.80%	20.50%	2.80%	15.20%			
0.18	9.60%	20.90%	2.70%	15.40%			
0.18	9.40%	21.30%	2.70%	15.60%			
0.18	9.10%	21.70%	2.10%	15.80%			
0.18	3.00%	13.10%	-1.40%	8.50%			
0.18	2.70%	13.00%	-1.70%	8.40%			
0.18	2.50%	12.90%	-2.30%	8.30%			
0.18	2.30%	12.80%	-3.00%	8.10%			
0.18	5.80%	37.60%	3.40%	25.70%			
0.18	5.60%	37.50%	3.10%	25.70%			
0.18	5.30%	37.50%	3.10%	25.70%			
0.18	5.00%	37.40%	2.90%	25.60%			
0.18	7.70%	38.10%	32.60%	31.90%			
0.18	7.70%	38.20%	33.00%	32.10%			
0.18	7.10%	38.40%	33.30%	32.30%			
0.17	7.20%	24.20%	20.20%	20.80%			
0.17	7.30%	24.30%	20.50%	21.00%			
0.17	7.20%	24.20%	20.20%	20.80%			
0.17	7.30%	24.30%	20.50%	21.00%			
0.17	3.60%	18.30%	17.70%	16.10%			
0.17	3.30%	18.60%	17.90%	16.30%			
0.17	3.20%	18.70%	17.90%	16.40%			
0.17	3.20%	18.80%	18.10%	16.50%		Time-Varying Beta	
0.17	1.40%	-0.80%	3.70%	0.30%	Prime Retail	**Prime Luxury Residential**	Prime Office
0.17	1.20%	-0.80%	3.60%	0.20%	0.154522	1.258436	0.866193
0.17	1.00%	-0.80%	3.40%	0.10%	0.165456	1.261191	0.836673
0.16	0.70%	-0.90%	3.30%	0.00%	0.179787	1.263291	0.805842
0.16	-9.20%	-12.40%	-30.60%	-15.00%	0.258698	1.20773	0.940987
Office (SqM)	Prime Retail	Prime Luxury Residential	Prime Office	Singapore Real Estate Market	Prime Retail	Time-Varying Beta	Prime Office
0.16	-9.40%	-12.60%	-30.80%	-15.10%	0.298008	1.179639	1.014551
0.16	-9.70%	-12.70%	-30.80%	-15.20%	0.321587	1.161599	1.063463
0.16	-9.90%	-12.80%	-30.90%	-15.30%	0.336642	1.148918	1.09938
0.16	8.30%	3.80%	0.80%	3.90%	0.322483	1.152392	1.097767
0.16	8.30%	3.80%	0.40%	3.80%	0.312512	1.156651	1.091202
0.16	8.30%	3.80%	0.40%	3.80%	0.302927	1.160679	1.085046
0.16	8.30%	3.80%	0.30%	3.80%	0.293719	1.164545	1.079047
0.16	20.20%	8.90%	7.50%	10.10%	0.288991	1.166813	1.075073
0.16	20.30%	9.10%	7.60%	10.20%	0.290543	1.153749	1.116086

0.16	20.50%	9.20%	7.90%	10.30%	0.296403	1.137962	1.163858
0.16	20.70%	9.30%	8.20%	10.50%	0.307262	1.119115	1.21917
0.16	-2.10%	2.10%	-8.40%	-0.10%	0.328836	1.094832	1.288984
0.16	-2.10%	2.10%	-8.20%	0.00%	0.3572	1.082415	1.307996
0.16	-2.10%	2.10%	-8.30%	0.00%	0.392743	1.067212	1.331078
0.16	-2.10%	2.10%	-8.50%	-0.10%	0.443871	1.046395	1.360855
Office (SqM)	Prime Retail	Prime Luxury Residential	Prime Office	Singapore Real Estate Market	Prime Retail	Time-Varying Beta	Prime Office
0.18	2.80%	-2.80%	-4.20%	-1.90%			
0.18	3.10%	-2.50%	-3.90%	-1.60%			
0.18	3.40%	-2.10%	-3.10%	-1.20%			
0.18	9.80%	20.50%	2.80%	15.20%			
0.18	9.60%	20.90%	2.70%	15.40%			
0.18	9.40%	21.30%	2.70%	15.60%			
0.18	9.10%	21.70%	2.10%	15.80%			
0.18	3.00%	13.10%	-1.40%	8.50%			
0.18	2.70%	13.00%	-1.70%	8.40%			
0.18	2.50%	12.90%	-2.30%	8.30%			
0.18	2.30%	12.80%	-3.00%	8.10%			
0.18	5.80%	37.60%	3.40%	25.70%			
0.18	5.60%	37.50%	3.10%	25.70%			
0.18	5.30%	37.50%	3.10%	25.70%			
0.18	5.00%	37.40%	2.90%	25.60%			
0.18	7.70%	38.10%	32.60%	31.90%			