Journal of Business and Economics, ISSN 2155-7950, USA
March 2013, Volume 4, No. 3, pp. 230-241

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A Unified Paradigm of Interest-Rate Modeling

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Abstract: Prices of assets and their derivatives are congruent, yet practitioners price them separately, especially in the fixed-income market where quantitative models are constructed within disconnected ivory towers. This research proposes a *unified paradigm* for the term structure of interest rate: from a financial aspect, the model must provide a pricing mechanism for both the underlying assets and their derivatives; from a mathematical aspect, the model must show the internal consistency under different measures; from a computational aspect, the model must come equipped with a (pseudo) closed-form pricing formula or an efficient simulation method. Furthermore, "unified" indicates the importance of integrating empirical and theoretical results in term-structure modeling for cross-asset markets such as with interest-rate futures and commodity futures. Also, "unified" implies the flexibility of integrating various trading strategies and the novelty of bridging the complete and incomplete market assumptions.

Key words: Unified Paradigm; interest-rate model; bond model; risk management; derivatives pricing framework

JEL codes: G12, G13

1. Motivation

Interest-rate modeling has long been a building block for complex derivative pricing. Mainstream research has focused on interest-rate dynamics. In much of that research, risk preferences are indifferent under an artificial probability measure called the risk-neutral measure. With this, a collection of pricing formulae are constructed based on the market completeness assumption. Despite the elegance of risk neutrality, and after studying deeply its practicality in applications, one notices that a crucial yet natural factor is missing: trading strategies. Two important difficulties emerge when incorporating trading strategies into a model: prices are no longer uniquely defined, and the derivatives are only hedgable through proper tools. Therefore, this research establishes a framework for modern pricing theory that overcomes this limitation in current interest-rate models.

In dissertation, three real-world situations are utilized. These are the problems of liability-driven investing (LDI), asset valuation and derivative pricing, and the dichotomy of the preferred-habitat hypothesis of economists with the local expectations hypothesis favored by financial engineers (about these, much more will be said later). These situations will help motivate certain questions and provide a setting to illustrate important and novel results which answer these questions.

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1.1 Liability-driven Management

Retirement plan is a major liability for corporate and public institutions. When a plan is calculated by a predetermined formula based on employee's future benefit, it is a *defined benefit (DB)* pension plan. Under this plan, as employees retire, the employers have the obligation to fulfill employees' benefits; therefore, employers bear the investment risks. When a plan is set up by individual accounts for participants and the benefits are calculated based on the amounts credited to these accounts plus investment earnings, it is a *defined contribution (DC)* pension plan. Only the contributions which employers put into the account are guaranteed, but the investment portion may fluctuate. Therefore, employers take no investment risks. In current days, despite pension-fund management strategies have been extensively studied, many plans are in a state of crisis due to improper asset management. Many issues have been discussed in details in Waring (2011). Consequently it is still challenging for the pension plan world to break new ground on managing risks. In this research we seek a better modeling tool for quantifying risks especially for DB plans, and design new financial instruments for DB practitioners.

In 2006, the *Pension Protection Act (PPA)* became US law. Among other objectives, the PPA regulates the present-valuing methodology for the computation of the pension liabilities (associated with a sponsor's defined benefit plan) by introducing a curve called the *High-Quality-Market (HQM) Corporate Bond Yield Curve*, a curve based on the universe consisting of commercial paper and single A-rated and better corporate bonds. It serves the purpose not only of measuring the pension liability on the balance sheet, but also for establishing a standard benchmark for pension fund managers. The curve is constructed using five splines constructed from a linear combination of eight pre-defined degree-three B-Splines at knots 0, 1.5, 3, 7, 15, and 30 years. The curve has zero second derivative at the short end, and zero first derivative at year 30 and beyond. The 30-year forward rate is constrained to be the average forward rate in the 15-to-30-year range. The curve is constructed by the least squares optimization between the market and model bond prices, and the weighting scheme is adjusted by bond par amounts and durations. For more details, please see Girola (2007).

Despite legislative efforts, the total shortfall of corporate pension funds has topped $260 billion in 2010 (Reilly, 2010). Since the 2008 financial crisis, pension fund sponsors have decreased the portion of risky assets in their pension portfolios to adjust their investing objectives and reduce underfunded risk. However, the holes are too large to fill since over 70% of the funds are already underfunded. Though what has been done has been done, the question is whether we can do better in the future. The answer is positive: creating a dedicated bond benchmark-portfolio[1] to fulfill pension liability payments is one approach, and designing new hedging tools based on the pension liability is another.

For illustrating the first solution, assume we are given a pension liability cash flow, and the present value of the liability fluctuates as interest rates and bond prices move. Since the future values of liabilities are certain, fixed-income asset portfolios are attractive since they can be made to match liability cash flow and yet hedge corporate funding risk. The duration-matching method is the most popular approach for this class of LDI problem due to its simplicity. A pension liability can be decomposed into several duration buckets. Then portfolio managers can fill the buckets with the targeted bonds within the same duration ranges, and managers' performances are easily compared to bond indices such as the *Barclays Aggregate*. However, duration matching, which only captures the parallel-shifting of interest-rate curve movements, can only partially hedges interest risk.

[1] Many tools and approaches have been around for a long time, but they were ignored due to improper asset management.

There is also no guarantee that the liability payments can be fulfilled. Therefore a better way to formulate this problem might be to design a dedicated portfolio.

Dedicated portfolios (Tutuncu & Cornuejols, 2007) or (Dynkin et al., 2006) neutralize interest-rate risk by matching asset and liability structures. Such an asset portfolio is a liability-driven portfolio since its asset allocation decision is based on the liability cash flow. Denote a liability profile by $\pi_L = \left(\tau^L, c^L\right)$ where $\tau = \left\langle \tau_1, \tau_2, \ldots, \tau_M \right\rangle$ is the payment schedule in units of decimal years and $c = \left\langle c_1, c_2, \ldots, c_M \right\rangle$ is the corresponded dollar amount. Similarly we can define an asset portfolio π_P. Then a liability-driven asset portfolio can be characterized by the following optimization program:

$$\min_P \sum_{i=1}^{M} w_i \left| c_i^L - c_i^P \right| \tag{1}$$

subject to

$$\sum_{i=1}^{m} c_i^L \le \sum_{i=1}^{m} c_i^P, \quad m = 1, 2, \ldots, M-1 \tag{2}$$

and

$$\sum_{i=1}^{M} c_i^L = \sum_{i=1}^{M} c_i^P \tag{3}$$

Where $w = \left\langle w_1, w_2, \ldots, w_M \right\rangle$ is a pre-defined weighting scheme.

Soon after constructing such a primitive dedication strategy we recognize problems beyond naive cash flow matching. There are three additional major requirements that need to be addressed. First, credit risk plays a significant role because the corporate bond yield curve is heavily determined by the credit distribution, which is determined by the total-par-amount percentages of credit sectors in a dataset. If the liability-driven asset portfolio has an inconsistent credit distribution, we could end up tracking an incorrect discount curve. Secondly, the relatively few corporate bonds with over-10-year maturity have poor liquidity, thus the bid-ask spreads are expected to widen with increasing demand for long-maturity corporate bonds. This problem suggests we could use US Treasury bonds/notes/bills and US Treasury strips to reduce the liquidity risk. However, this strategy costs more and incurs more credit risk. Thirdly, pension fund sponsors might not be willing to contribute more than the legal-required amount, which means that an asset portfolio value must be less than or equal to the liability present value. This is in conflict not only with the possible higher cost if Treasury securities required reducing liquidity risk, but also with the goal of a dedicated portfolio if no feasible solution exists under this budget.

We have conducted a series of empirical investigations demonstrating that we have been able to successfully construct a liability-driven asset portfolio satisfying all the above-mentioned requirements. In these investigations, the asset portfolio and liability present value have a correlation[2] exceeding \$0.98, but the relationship is not perfect. As a result two interesting questions emerge. First, the imperfect correlation indicates the possibility for profits if portfolio managers can beat the market. The question here is whether we can turn this passive pension fund strategy into an active alpha generating strategy. The answer turns out to be positive, but this issue is not part of this research and will be discussed elsewhere (Lin & Audley, 2012). The other more intriguing question is how can we quantify and hedge this "unhedgeable risk". We illustrate this unhedgable risk in the following. When a

[2] Formal definition of tracking error can be found in Amenc & Le Sourd, 2003.

pension liability is discounted by the corporate bond yield curve, the discounted liability value represents the value of a "virtual" bond portfolio consisting of commercial paper and single A and above rated corporate bonds, and downgrades to below A—never occur—those bonds vanish on downgrade. On the other hand, any bond in a real asset portfolio could be downgraded, cheapen, and sold at a disadvantaged price. The only way to mitigate the drop-off effect is to increase the rebalancing frequency, which is costly. So, new derivatives are necessary for developing prudent strategies. One solution is a credit default swap contract to provide monthly or quarterly compensation for unexpected downgrades. The other solution is a liability-driven index option. It is the latter that stimulated our new thinking about interest-rate models.

1.2 Assets Evaluation and Derivatives Pricing

In the previous section we have seen an actuarial application drawn from quantitative portfolio strategies and risk assessment. To quantify the *ex ante* underfunded risk we needed an interest-rate model. On the other hand, to use new derivatives for hedging we also need an interest-rate model for pricing. Because the two tasks are performed under two different probability measures—the real-world measure and the risk-neutral measure, a critical question is "how do we coherently switch a particular model between the measures?" Before we dig into this problem, we provide three situations in which our model aims to solve the difficulties at hand:

- You want to borrow $4M in 3 months, $3M in 6 months, $2M in 9 months, and $1M in 12 months according to LIBOR rates. For hedging the interest rate risk, you also enter into forward LIBOR cap contracts with matching maturities and principal amounts. Can you use an interest-rate model to calculate the expected weighted average borrowing cost and the expected profit-and-loss (P & L) of your derivatives position one month later?
- You have a portfolio containing short positions in Treasury bond futures for $100M, Treasury note futures for $100M, and Treasury bill futures for another $100M. Each position has futures contracts in various maturities. At the end of each month, for locking in the P&L in all the positions maturing in 3 months you enter into Treasury bond/note/bill futures options with matching maturities and principal amounts. Can you use an interest-rate model to calculate the value at risk (VaR) of the futures portfolio and the expected cost of hedging when you are in the middle of each month?
- You invest $10M in AA-rated corporate bonds, and some of the bonds are callable bonds. Expecting that the credit spread will widen, you also underwrite $1M AA credit spread put options with various strikes. Can you use an interest-rate model to calculate the option prices and calculate the expected annual return of the bond portfolio?

Without specifying the structure of risk premiums exogenously, the answer is usually negative because the model requirement of internal consistency often leads us to some unreasonable assumptions. This means that interest-rate models satisfying the usual no-arbitrage conditions under the risk-neutral measure are often too degenerate to switch back to the real-world measure. This motivates us to seek a better market and model setting such that assets and their derivatives can be appropriately priced. This leads us to the preferred-habitat hypothesis (Fabozzi, 2002) discussed in the following subsection.

1.3 Preferred-Habitat Hypothesis

Traditionally, in either equilibrium or no-arbitrage term structure models, the no-arbitrage condition follows the local expectation hypothesis, which shows that every bond is treated as a derivative of the short rate[3]. Because of internal consistency, this bond market is complete. Nonetheless, several recent studies have challenged these

[3] An introduction between interest-rate models and no-arbitrage conditions is given in Jarrow, 2009.

assumptions and shown the contrary evidence including: idiosyncratic risk, market segmentation, and supply-and-demand factors. In short, market completeness is too good to be true, and we aim to resolve the problems mentioned above by a more suitable hypothesis. We new review those studies.

In 2010, Anderen & Benzoni (2010) showed that affine term structure models are not sufficient to model an arbitrage-free bond market. Their argument showed that systematic bond volatility is not merely a span of volatilities from the cross-section of yield volatilities. From this they showed that the Treasury market is incomplete since yield volatility risk cannot be hedged solely through Treasury securities. Moreover, empirical research often shows a contradiction with theoretical hedging strategies under the market completeness assumption. For example, the completeness assumption would say that if $N=3$ and 5, 10, 15-year time-to-maturity bonds are chosen, the chosen bonds can be used to hedge all other bonds including those between 0 to 5-year bonds and 15 to 30-year bonds. This hedging strategy is not effective in practice.

In addition, no-arbitrage models do not capture supply and demand in the Treasury bond market. In 2010, Krishnamurthy & Vissing-Jorgensen (2010), showed that changes in Treasury supply have large effects on a variety of yield curve movements. Though their research focused on the spread between corporate and Treasury bonds, the result can be further applied to connect the money market account to the bond market. Furthermore, Duffie (1996) exposed the idiosyncratic risk in short-term Treasury bills and postulated that this risk was the consequence of the time-variation of the convenience yield[4] for holding Treasury bills. Concurrent research conducted by Duffie (1996) showed similar idiosyncratic supply and demand for different maturities in the repo market. Moreover, we expect a similar result between deliverable Treasury bonds and Treasury bond futures. Here the cheapest-to-deliver option may affect bond market equilibrium through an imbalance of supply and demand in different maturities. Those effects which are not driven by systematic factors can be modeled as the convenience yield, and we thereby extend the name “convenience yield” to bond markets. In some sense we should correctly state them as “inconvenience yield” since supply-and-demand constraints are posed on the bonds.

This motivates us to adopt the preferred-habitat hypothesis in the bond market without the usual assumption on completeness.

2. A Unified Paradigm

The models previously described all have their advantages and disadvantages. In what follows, we shall focus on retaining the advantages of each model and resolving the drawbacks. Preliminary research indicates that next-generation models must not only have the ability to model non-negative interest rates, but most also satisfy the major criteria in finance, mathematics, and computation described below.

We adopt the name “unified” to emphasize that a next-generation model must survive the tests from interdisciplinary areas and shall exist under both the real-world and an equivalent-martingale measure. Furthermore, “unified” indicates the importance of integrating empirical research into theoretical term structure models as well as the utilization of interest-rate models to enable cross market research such as with the fixed-income futures market or commodity futures market. Lastly, “unified” implies the flexibility of integrating various trading strategies so the pricing results are more accurate. In the last section, we are going to demonstrate

[4] Convenience yields were defined for commodities due to the cost of storages, see German, 2005. We analogously adopt this concept for idiosyncratic risk in fixed-income markets.

how to build such a unified model.

2.1 Model Criteria

2.1.1 Finance

In finance, the model must be able to price the underlying asset and its derivatives simultaneously. We believe that the market of a particular asset and its derivatives are congruent, and there is no reason to price them separately. For example, we want to construct an interest-rate model for the LIBOR-swap rate curve under the real-world measure and use it to price LIBOR derivatives such as LIBOR caps, floors, or even serial options under an equivalent-martingale measure. Another example is the valuation of bullet and callable bond structures. A callable bond is a bullet bond with an embedded American style call option owned by the issuer; however the bullet bond price is determined under the real-world measure and the embedded option may be priced under an equivalent-martingale measure. Therefore, the next-generation model must be able to value a callable bond by valuing the bullet bond and the embedded American style call option simultaneously. As a further example, let's consider the case where the underlying assets can be fixed-income futures contracts. Both Treasury bond futures and Eurodollar futures markets have been growing rapidly for a considerable time, and options, such as Treasury bond futures options and Eurodollar futures options, are also traded in the OTC market. Unlike equity futures or exchange rate futures, there is an interest-rate term structure associated with the fixed-income futures market. Thus, fixed-income futures are more complicated. And, as an interest-rate model is designed to price fixed-income futures, a next-generation model can also be used to price futures options. Many interest-rate models are utilized in disconnected fashion (even in the same market), and it is our first goal to resolve this situation.

Modern financial markets create not only diverse traded assets but also a variety of trading environments, such as high-frequency trading or model arbitrage trading. As new market equilibrium is created, we observe that investors with different investing purposes have different trading patterns, and we expect a next-generation model to incorporate this phenomenon. This consideration is outside of the Markowitz framework where financial securities are grouped only by their means and variances. Therefore, in the derivatives pricing where the risk preferences are taken from the risk-adjusted equation, we propose a new methodology to bridge the gaps between investors whose evaluation are affected by their trading strategies. A next-generation model shall be joined with both the dynamic and static hedging pricing formulae as described below.

2.1.2 Mathematics

In mathematics, the model must show equivalency under different probability measures by explicitly characterizing the market prices of risk endogenously. Market price of risk can be thought of as just another name for the risk-premium, but mathematically it represents an adapted process with respect to the filtration generated by the source of uncertainty. If the sources of uncertainty are Brownian motions, then we can apply the Girsanov Theorem to change the probability measures from the real-world measure to an equivalent-martingale measure. Furthermore, closed-form solutions are essential for dynamic hedging and so that exotic derivatives can be fairly hedged despite the fact that the difficulty to do this increases when the interest-rate volatilities are stochastic (see Chapter 6).

Pricing formulae need to be defined and justified rigorously according to different hedging settings when the market is not complete. To do this, an equivalent-martingale measure needs to be extended to incorporate trading strategies. Then the link between the complete and incomplete market settings must be addressed, and the next-generation model will incorporate a mechanism for these markets to operate interchangeably. We emphasize that market completeness does not depend on *a priori* the existence of a complete set of traded assets, but rather on the existence of an entity that can make the market in those assets if an arbitrage opportunity emerges. For example a fixed-income bullet bond can be created by being long a Treasury bond/note/bill and short a credit default swap contract. Hence, an incomplete market setting does not rule out the possibility of becoming complete as long as minimal regulations are imposed and maximal financial innovations are utilized. The mathematical component, as a result, is the building block for a paradigm shift.

2.1.3 Computation

In computation, we ask if the model can be simulated by the same algorithm efficiently under different measures. This issue is more important in interest-rate modeling since there is a tradeoff between satisfying the mathematical component and computational component, and finding a model that satisfies both is not trivial. The details of this are discussed in the next section when we introduce the no-arbitrage conditions. We emphasize that the simulation method must work simultaneously under different probability measures. Moreover, as American-style options can be priced by recombination tree algorithms, we specifically use the word “efficiently” to implicitly indicate that a next-generation model should also be simulated by recombination trees.

Finally, the model should admit calibration from a variety of market constituents. For instance, exotic derivatives are priced based on plain-vanilla derivatives, so a closed-form caplet/swaption pricing formula which connects to the market implied volatilities is essential. As a consequence, providing the flexibility to tune the model for more complex products or more fundamental assets is one of the major considerations for computation.

2.2 Market Settings

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ denote the filtered probability space. Let *F(t, T)* denote the instantaneous default-free interest-rate curve dynamics at time *t* with time-to-maturity (term) *T*. Let

$$B(t,s) = \exp\left(-\int_0^{s-t} F(t,u)du\right)$$

denote the curve-implied bond price dynamics, which is the conventional representation for bond modeling. We shall examine this math form and extend the framework to a more flexible market setting.

2.2.1 Default-Free Bond Market

Let the US Treasury be the only issuer, and let the default-free securities refer to US Treasury bonds/notes/bills. Many have pointed out the idiosyncratic movements in the US Treasury bond yield curve in isolated maturity ranges. The reasons are mostly associated with US Treasury's actions for specific fiscal policy issues such as budget deficits or the Federal Reserve's monetary policy actions. Idiosyncratic risk forces hedgers (or preferred-habitat investors) to hedge interest-rate risk only by purchasing bonds with specific maturities.

Assume that *hedgers* are preferred-habitat investors. For example, pension funds focus on long-term bonds. To simplify the analysis, we assume each hedger looks for several specific bonds for hedging; then he buys and holds the bonds until maturity. Given a fixed future liability cash flow, the hedging cost is simply the sum of current bond prices. Since the instantaneous forward-rate curve *F(t, T)* is generated from the default-free bond yields, which are assumed to be the constant reinvesting rate until the bond maturity, this curve represents the total return of “risk-free” assets as buying-and-holding default-free bonds until its maturity is risk-free.

With our model setting so far, hedgers segment the market, and the US Treasury bond market would respond in different maturity ranges without a proper market mechanism. Therefore, we need to introduce two additional kinds of traders: arbitrageurs and speculators.

Arbitrageurs look for mispriced securities. Their action implies the existence of false or incomplete information embedded in the current interest-rate curve, but corrections are expected to be made in the near future. In other words, if the arbitrageur's evaluation is correct, they expect near-term profit from yield curve trading. The trades usually have zero up-front cost. Beyond the role of arbitrages to "correct" mispriced securities and make market information transparent, their participation further integrates the bond market. That means, if the bond market is complete, then cross-maturity hedging strategies are feasible. However, we shall see that it is not the case as we introduce the speculators.

Unlike hedgers, *speculators* seek higher returns by taking interest-rate risk. That means a speculator expects that total returns need to be compensated proportional to interest-rate risk (or bond volatilities). From a speculator's point of view, bond dynamics can be derived from the Ito Lemma, i.e.,

$$B(t,s) = B(0,s)\exp\left(\int_0^t F(u,s-u)du - \int_0^t\int_0^{s-u}\mu(u,v)dvdu - \int_0^t\int_0^{s-u}\sigma(u,v)dvdW(u)\right) \tag{4}$$

Where μ and σ are instantaneous drift and volatility terms respectively. Here W represents the standard Brownian motion under the real-world probability measure P. Thus, for speculators the expected bond price at time t is

$$\mathrm{E}\left[B(t,s)\right] = B(0,s)\mathrm{E}\left[\exp\left(\int_0^t F(u,s-u)du - \int_0^t\int_0^{s-u}\mu(u,v)dvdu + \frac{1}{2}\int_0^t\left(\int_0^{s-u}\sigma(u,v)dv\right)^2 du\right)\right] \tag{5}$$

Jensen's inequality shows

$$\mathrm{E}\left[B(t,s)\right] = \mathrm{E}\left[\exp\left(-\int_0^{s-t}F(t,u)du\right)\right] \geq \exp\left(\mathrm{E}\left[-\int_0^{s-t}F(t,u)du\right]\right) \tag{6}$$

Which means that, under the real-world measure, the speculators' expected return is no less then hedgers' expected return on the same bond over the same period of time. From a speculator's point of view, the "default-free" forward-rate curve is, as a result, higher for what would be their "risky" strategies.

Speculators take risks and pursue higher returns. For example, if a bond is expected to experience a lower price, a speculator short sells it. Unlike hedgers or arbitrageurs, a speculator expects to make a profit by taking risks; therefore the expected return of the position depends not only on the current bond yields but also on bond volatilities which are the usual measures for interest-rate risks. We don't assume the bond market to be complete; thus we might have multiple equivalent martingale measures, and the expected rates of return do not necessarily equal to the short rate. We formulate this observation in the following section.

3. No-arbitrage Framework

A paradox is discovered in (Lin, 2010) that a default-free bond return cannot simply be valued by a conventional risk-neutral pricing formula. Lin claims that the contingent claim valuation is affected by trading strategies. To see this, let us assume a bond is dynamically hedgable under the HJM framework. Let the zero-coupon bond dynamic *B(t,s)* with maturity *s* at time *t* under the risk-neutral measure be written as

$$dB(t,s) = R(t)B(t,s)dt - \left(\int_t^s \sigma(t,u)du\right)B(t,s)d\tilde{W}(t) \tag{7}$$

Where R is the short-rate process, σ is the instantaneous interest-rate volatility, and $\tilde{W}$ is a standard Brownian motion under the risk-neutral measure $\tilde{P}$. Notice that the zero-coupon bond itself derived from the instantaneous forward-rate curve is default-free. Therefore

$$B(t,s)=B(0,s)\exp\left(\int_0^t R(u)du-\tfrac{1}{2}\int_0^t\left(\int_v^s\sigma(v,s)dv\right)^2 du-\int_0^t\left(\int_v^s\sigma(v,s)dv\right)d\tilde{W}(u)\right) \tag{8}$$

Which shows

$$\frac{1}{B(0,s)}=\frac{B(s,s)}{B(0,s)}=\exp\left(\int_0^t R(u)du-\tfrac{1}{2}\int_0^t\left(\int_v^s\sigma(v,s)dv\right)^2 du-\int_0^t\left(\int_v^s\sigma(v,s)dv\right)d\tilde{W}(u)\right) \tag{9}$$

The paradox of Equation (9) is the consequence of matching the known bond price on the left-hand side to a random variable on the right-hand side. The phenomenon can be explained by separating the market participants. Under the usual market settings, market participants only refer to traders who hold their positions over relatively short amount of time. Their profits are mostly contributed from market movements modeled as bond volatilities. Their compensation per unit volatility over the risk-free rate is described by the market price of risk which connects the real-world and the risk-neutral measures. Nonetheless, there is another group of market participants who seek hedging tools. They tend to buy and hold the bonds until maturity. If bonds are default-free, then the expected rates of return are known when the positions are created. Consequently the left-hand side of the equation represents this type of market participants—the hedgers.

To resolve this issue, this research proposes a solution by introducing an inconvenience-yield term structure associated with default-free bonds. Before formulating this new terminology, we first need to examine trading strategies from which we are able to calculate risk-free returns from the default-free interest-rate term structure. When $T = 0$, F(t,0) represents the "default-free" short rate which coincides with the "risk-free" short rate since no default risk is involved in F(t,0) when time-to-maturity is zero. Nonetheless, both a risk-free bond and a default-free bond have the same terminal value, and the dynamic-hedging valuation equation yields the same value for the two different bonds. Therefore in this research we propose a new market/model setting for these two instruments and identify the subtle differences between risk-free and default-free bond term structures.

3.1 Interest-Rate Curve Dynamics

First, we have to relate the default-free interest-rate curve to a risk-free curve. Let $\hat{F}(t,T)$ be the instantaneous risk-free forward-rate curve, and let $\hat{B}(t,s)$ denote the risk-free bond with maturity *s*. Since $\hat{B}$ is a risk-free asset, its stochastic differential form

$$d\hat{B}(t,s)=R(t)dt \tag{10}$$

has zero volatility term, which implies that the *s-forward risk-neutral measure* $\hat{\mathrm{P}}$ *with respect to the risk-free bond* $\hat{B}(t,s)$ is the same as the risk-neutral measure $\hat{\mathrm{P}}$. Therefore the instantaneous forward-rate dynamics satisfy

$$\hat{F}(0,t)=\hat{\mathrm{E}}^t\left[\hat{F}(t,0)\right]=\hat{\mathrm{E}}\left[\hat{F}(t,0)\right] \tag{11}$$

and the risk-free curve $\hat{F}$ at time t constructed from the short rate $\hat{F}(t,0)$ is

$$\hat{F}(t,T)=-\frac{\partial}{\partial T}\ln\left(\tilde{\mathrm{E}}\left[\exp\left(-\int_t^{t+T}\hat{F}(u,0)du\right)\middle|\mathscr{F}_t\right]\right) \tag{12}$$

Here we emphasize again that the default-free curve must coincide with the risk-free curve when $T = 0$, i.e.,

$$\hat{F}(t,0)=F(t,0)=R(t) \tag{13}$$

Hence we also have

$$\hat{F}(t,T) = -\frac{\partial}{\partial T}\ln\left(\tilde{\mathrm{E}}\left[\exp\left(-\int_t^{t+T} R(u)du\right)\middle|\mathscr{F}_t\right]\right) \tag{14}$$

Risk-free bonds are not traded assets. In fact, risk-free instruments shall have zero volatility (in differential form) while traded assets have non-zero volatility. Nonetheless, risk-free instruments can be created from risky assets. Here we compare the term-structure of risk-free bonds derived in Equation (14) to the term structure of the risk-free instruments created by imposing special trading strategies.

From Equations (13) and (14), the risk-free zero-coupon bond $\hat{B}$ at time *t* with maturity *s* is written by

$$\hat{B}(t,s) = \tilde{\mathrm{E}}\left[\exp\left(-\int_t^s F(u,0)du\right)\middle|\mathscr{F}_t\right] = \exp\left(-\int_0^{s-t}\hat{F}(t,u)du\right) \tag{15}$$

Which is synthetically created and valued under a market without trading frictions and without supply-and-demand factors. Therefore $\hat{B}$ is mainly used it as a reference set in this research, and it cannot be used for calculating the risk-free return in the real-world market. The actual risk-free return, which needs to be calculated based on feasible assets, is measured based on buying and holding a default-free bond until its maturity. This risk-free return over the time period $[t,s]$ is written as

$$\frac{1}{B(t,s)} = \exp\left(\int_0^{s-t} F(t,u)du\right) \tag{16}$$

The expected risk-free return is then calculated by

$$\exp\left(\tilde{\mathrm{E}}\left[\int_0^{s-t} F(t,u)du\right]\right) \tag{17}$$

Which is less than or equal to a default-free return over the same time period

$$\tilde{\mathrm{E}}\left[\exp\left(\int_0^{s-t} F(t,u)du\right)\right] \tag{18}$$

The key of this research is to characterize the difference when comparing to a risk-free bond $\hat{B}$, which is created freely only in theory. Hence there shall exist an *inconvenience yield* C associated with actual risk-free return to prevent arbitrages such that

$$B(t,s) = \tilde{\mathrm{E}}\left[\exp\left(-\int_t^s F(u,0) + C(u)du\right)\right] \tag{19}$$

Following from the observation, we now state the no-arbitrage condition under the unified paradigm. Let $R(t;s)$ denote a *s-forward-start short rate* implied by *F*(t,0) under a risk-neutral measure, from which we write

$$dR(t;s) = \mu(t,R(t;s))dt + \sigma(t,R(t;s))d\tilde{W}(t;s) \tag{20}$$

Where $s \geq 0$ is the starting time and $\tilde{W}(t;s) = \tilde{W}(t+s)$, and

$$R(0;s) = \tilde{\mathrm{E}}\left[F(s,0)\right] \tag{21}$$

Hence, to avoid arbitrages, we shall have that for all $s \geq 0$

$$-\frac{\partial}{\partial T}\ln\tilde{\mathrm{E}}\left[\exp\left(-\int_0^T R(u;s) + C(u)du\right)\right] = \tilde{\mathrm{E}}\left[F(s,T)\right] \tag{22}$$

The condition above indicates that, for any future time *s* ≥ 0, the future term structure implied by the forward-start short rate *R* plus a (deterministic) inconvenience yield parameter *C* is equal to the expected default-free term-structure.

The inconvenience-yield term structure *C* might not be deterministic or unique. Consequently the risk premium cannot be quantified uniquely. Addressing on this concern, we can assume *C*(*T*) to be a deterministic function, and then a unique risk-neutral measure $\tilde{P}$ can be defined through a proper no-arbitrage condition.

3.2 Bond-Price Dynamics

It is necessary to have an inconvenience yield so as to have an arbitrage-free market. Also, the incomplete market setting implies the existence of unhedgable idiosyncratic risk within each bond. By incorporating Brownian sheet, interest-rate models under this framework are consolidated by introducing the idiosyncratic factor associated with each bond. To view that framework under a modified probability space, let $(\Omega, \mathscr{H}, \mathscr{H}_t, P)$ denote the filtered probability space when idiosyncratic risks are included. By carefully shaping the correlation structure in any given Brownian sheet, a unique equivalent martingale measure $\ddot{P}$ exists where

$$\frac{dB(t,s)}{B(t,s)} = \ddot{\mathrm{E}}\left[F(t,s-t)\right]dt - \left(\int_0^{s-t}\sigma(t,u)du\right)d\tilde{W}(t) - d\tilde{W}(t,s-t) \tag{23}$$

Where $\tilde{W}(t,T)$ is a standard Brownian sheet under the measure $\ddot{P}$. Consequently, the expected bond price can be written as

$$\ddot{\mathrm{E}}\left[B(t,s)\right] = \exp\left(-\int_0^{s-t}\ddot{\mathrm{E}}\left[F(t,u)\right]du\right) = \ddot{\mathrm{E}}\left[\exp\left(-\int_t^s R(u;t) + C(u-t)du\right)\right] \tag{24}$$

Furthermore, for derivatives pricing purpose we write

$$\begin{aligned} &\ddot{\mathrm{E}}\left[\exp\left(-\int_0^s F(u,0)du\right)g\left(X(s)\right)\right] \\ &= \exp\left(\int_0^s C(u)du\right)\ddot{\mathrm{E}}\left[\exp\left(-\int_0^s F(u,0)+C(u)du\right)g\left(X(s)\right)\right] \\ &= B(0,s)\exp\left(\int_0^s C(u)du\right)\ddot{\mathrm{E}}^s\left[g\left(X(s)\right)\right] \end{aligned} \tag{25}$$

Where $\ddot{P}^s$ is called the *s-forward risk-neutral measure* in most literature where the zero-coupon bond $B(t,s)$ is used as the numeraire.

Although the prices are unique under this setting, we have to emphasize that this new framework only guarantees a fictitious complete market for exotic interest-rate derivatives. It is so because static hedging requires a liquid derivatives market and dynamic hedging requires an active bond-volatility market, and we have neither for exotic interest-rate derivatives. As a result, we have to emphasize that the fairness of a price derived under the $\ddot{P}$ measure relies on our claim that a complete market depends not only on the functionally traded assets, but also on the ability of the environment to create new contingent claims when the arbitrage opportunities are significant. In other words, we accept the prices not because they are fair, but because they are fair enough.

4. Summary

The new modeling paradigm is laid out in this paper: we start by stating the core criteria from three aspects (finance, mathematics, and computing) for a new conceptual model called the *unified paradigm*. We reshape the no-arbitrage conditions to accommodate this phenomenon where the market-setting may be incomplete. Unlike other research where models are conducted under the assumption of market completeness, we start this research by replacing that assumption with a weaker one. The default-free bond market under our new assumption is incomplete, yet this is more general for describing supply-and-demand factors or idiosyncratic risks associated with individual bonds. The convenience-yield idea is then, for the first time, introduced formally for a default-free bond market. It is significant and novel that we can bridge an interest-rate model built under an incomplete-market assumption to a complete-market framework which is the crucial pricing environment adopted by practitioners.

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