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Historic Investigation of Legendre's Proof about the 5th Postulate of "Elements" for Reeducation of Mathematics Teacher

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Abstract: We investigate the history of trials in proving the fifth axiom in "Elements" in order to develop a teaching material for high school mathematics. Arguments in "Elements" themselves are very clear, but the fifth axiom is apparently too complex as compared to the other four axioms. Here, we show the first to fifth axioms.

Key words: reeducation of teacher, Legendre's Proof, 5th Postulate elements

1. Introduction

Since "Elements" has published, many mathematicians had tried to prove the fifth axiom using the first to fourth axioms as prerequisite. Procles, the author of notes in "Elements" the first volume, published his own proof of it after describing a trail proof by Ptolemaios. By contrast, Euclid (330?-270?B.C.) described that ones that never cross as parallel. The question, "why the fifth axiom was needed in 'Elements'" was passed to Arabia. Simultaneously so did the effort to prove the fifth axiom to Arabic mathematicians.

A part of proof was imported to Europe. In Europe, as mathematics progresses, many mathematicians tried to prove it. Among them, work by Girolamo Saccheri（1667–1733）and Johann Heinrich Lamber（1728–1777）is well known. Due to this effort, non-Euclid geometry, where axiom does not hold, was born. In such a historical transition of proving the fifth theorem in "Elements", we investigated a famous proof of the fifth axiom in 'Elements' by Euclid, as a teaching material in high school mathematics. We hope that we utilize the historical transition to math education. There is a trial in proving the fifth axiom starting from a proposition. So we investigate this proposition to begin with.

2. A Certain Proof

Proposition: If a line crosses one of parallel straight lines, then the lines crosses the others.

Proof: Because half lines BF and FG are half lines starting from point F, the distance between them can be prolonged to any extent. I.e., when point P moves along half line FB towards point B, the distance between point P and half line FG (the length of perpendicular line from point P to half line FG) can be as long as it can extend when point B moves apart from point F. Therefore, the distance eventually exceeds the distance between the parallel lines. This means that FG and CD cross.

In fact, this proof has two problems. The first problem is around "when point P moves along half line FB

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towards point B, the distance between point P and half line FG (length of perpendicular line from point P to half line FG) can be any length as point B moves apart from point F". It is clear that the length of perpendicular line from point P to half line FG increases, but it is not clear that the length diverges to infinity and it needs a proof. Yet another problem is the statement "distance between parallel lines is constant" uses the fifth axiom, which is the proposition that needs to be proven. This is so called tautology.

Obviously, the definition of parallel lines does not include the notion of constant distance between parallel lines. In order to prove the constant distance, the fifth axiom becomes unavoidably necessary. For such a point, we think there is a need to investigate on how to conduct a lecture, but we feel that this can be used as a live teaching material of math history for high school students. On the other hand, once the proposition of Procles is proven, then the proof of the fifth axiom becomes a trivial task.

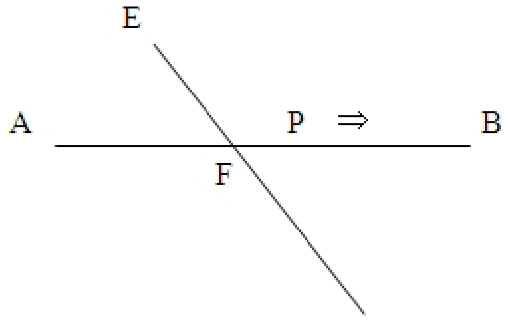

Figure 1 If a Line Crosses One of Parallel Straight Lines, then the Lines Crosses the Others

3. Historical Transition of the Proof by Legendre

Here, we investigate the proof of the fifth axiom by Lagrange in early 19th century. Adrien Marie Legendre (1752–1833) published a geometry textbook, Éléments de géométrie, in 1794. Since then, new editions have been published and the 15th edition was published in 1866. The text book by Legendre became the standard in geometry textbook in France. The book by Legendre is also famous by a side story that Galois read it and was influenced by it. This textbook is also translated in U.S. and published as a geometry textbook. Also in Japan, the book was used as a geometry textbook in early Meiji era. The book by Legendre looks like a best seller. The geometry textbook by Legendre publishes the proof of the fifth axiom in each edition. Although this book was sold well, he repeated to give a new proof whenever he realizes a mistake in his own proof. This means that Legendre was trying to prove it while Gauss already found a possibility of non Euclid geometry. Legendre published his own proof up to the last edition. Here, we investigate on one of the published proof. Also, if we assume the fifth axiom is true, then we are able to prove that sum of the internal angles of a triangle is twice of right-angle. Legendre started his proof from the opposite of this holds, in other words, that sum of the internal angles of a triangle is equal to twice of right angle and the fifth axiom are equivalent.

4. Proof by Legendre

Proposition: If sum of the internal angles of any triangle is twice of right angle, the fifth axiom holds.

Proof: Assume $\angle BEF + \angle DFE < 2R$.

Draw straight line FG from point F so that $\angle BEF = \angle EFG = 2R$ holds as shown below. Further, draw a segment of line from F to any point N on AB.

Next, choose point M on straight line AB so that MN = FM. Then, triangle FEM is isosceles triangle and its angle $\angle$MFN = $\angle$MNF is a half of $\angle$FME. $\angle$MFN = (1/2)$\angle$FME・・・* here, we used that sum of the internal angles of a triangle is twice of right angle, i.e.,

because $\angle MFN + \angle MNF + \angle EMN = 2R$,

it leads to that

$\angle MFG$

$= \angle EFG - \angle EFM$

$= \angle EFG - (2R - \angle FEM - \angle FME)$

$= (2R - \angle FEM) - (2R - \angle FEM - \angle FME)$

$= \angle FME$.

Therefore, due to *,

$\angle NFG = \angle EFG - \angle MFN$

$= \angle FME - \angle MFN$

$= (1/2)\angle FME$

$= (1/2)\angle MFG$.

Next, if we choose point P on straight line AB so that FN＝NP holds, using the same arguments as above, we get

$\angle PEG$

$= (1/2)\angle NFG$

$= (1/4)\angle MFG$.

From here, using the arguments above, we are able choose point R on AB so that $\angle RFG < \angle DFG$. Then, half line FD goes through inside of triangle FER and therefore crosses a side of triangle EF.

Therefore, straight line FD crosses AB.

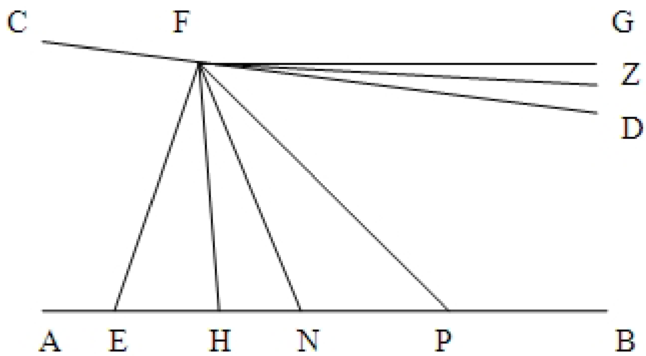

Figure 2 Sum of the Internal Angles of Any Triangle is Twice of Right Angle

From statements above, in order to prove the fifth axiom we need to prove that sum of the internal angles of a triangle is twice of right angle. To do that, Legendre proved the following proposition to start with.

Proposition: Sum of the internal angles of any triangle is equal to or less than twice of right angle.

Proof: Assume that sum of the internal angles of a triangle is always larger than twice of right angle. Extend side AC and draw a triangle so that triangle ABC ≡ triangle CDE. Repeat such a process for n times and draw triangles that are congruent to triangle ABC.

I.e., triangle ABC≡triangle CDE≡triangle EFG≡・・・≡triangle MNP.

Further, connect points B and D by a line segment. Similarly, connect points D and F and points F and H and so on. BD・・・NP becomes a broken line.

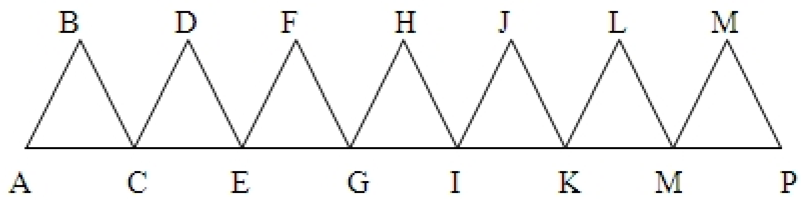

Figure 3 Sum of the Internal Angles of Any Triangle is Equal to or Less than Twice of Right Angle

Those points do not necessarily on a straight line.

Because triangle ABC≡triangle CDE,

we have $\angle BAC = \angle DCE$. Because ACE is a line segment,

$2R = \angle BCA + \angle BCD + \angle DCE$

$= \angle BCA + \angle BCD + \angle BAC$ holds.

On the other hand, from the assumption,

$2R < \angle BCA + \angle ABC + \angle BAC$ holds.

By compareing the above equalities,

$\angle ABC > \angle BCD$ holds.

Then, because

$AB = DC$, $BC = CB$, $\angle ABC > \angle DCB$ holds on triangles ABC and triangle DCB, side corresponding to the larger angle is longer due to ,'Elements' the first volume by Euclid, proposition 24.

I.e., $AC > BD$. Setting $AC - BD = \delta > 0$ gives $AP = nAC$. The length of broken line BD・・・LN is $(n-1)BD$. The length of broken line connecting points A and P, ABD・・・LNP is given as

$AB + (n-1)BD + NP$

$= AB + (n-1)BD + BC$

$= AB + BC - BD + nBD$

$= AB + BC - BD + nAC - n\delta$

$= AP + (AB + BC - BD - n\delta)$

Then, setting large enough n so that $AB + BC - BD - n\delta < 0$ will lead to that broken line connecting points A and P, the length of ABD・・・MOP, $AP + (AB + BC - BD - n\delta)$, is shorter than the length of line segment AP. Because the shortest broken line connecting two points is a line segment connecting those two point, this is contradiction. This contradiction was caused because we assumed that sum of the internal angles of triangle ABC is always larger than twice of right angle. Therefore, is was proven that sum of the internal angles of a triangle must be equal to or less than twice of right angle. We hope that we utilize this as a vivid historical teaching material of a proof by contradiction, which is hard to understand for high school students. Lastly, Legendre tried to prove the proposition below, but the proof includes mistake. Here we show the proof by Legendre.

Proposition: Sum of the internal angles of any triangle is twice of right angle.

Proof: Assume that sum of the internal angles of triangle ABC is always smaller than twice of right angle.

We define that sum of the internal angles of triangle ABC$=2R-\delta$, where $\delta > 0$.

We draw triangle DBC such that it shares side BC of triangle ABC and it is located to the opposite side of A so that triangle DBC≡triangle ABC. As shown in Figure 3, draw a straight line which passes on point D so that it crosses extention of AB at E and extention of AC at F. Here, we sum up all the internal angles of four small triangles, triangle DBE, triangle ABC, triangle BCD and triangle CDF.

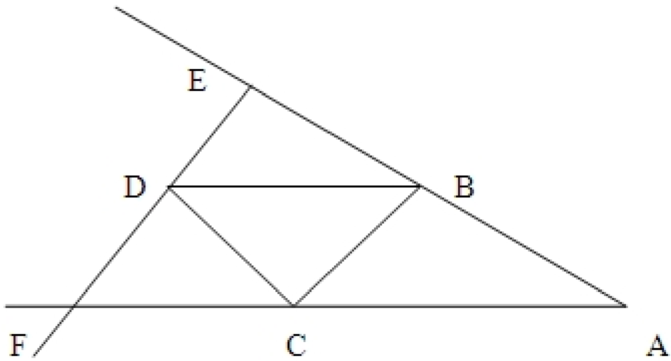

Figure 4 Sum of the Internal Angles of Any Triangle is Twice of Right Angle

Using the fact that ABE, ACF, and EDF are straight lines,

$$\angle BAC+\angle ABC+\angle BCA+\angle BCD+\angle CBD+\angle CDB+\angle DCF$$
$$+\angle CFD+\angle FDC+\angle BDE+\angle DBE+\angle BED$$
$$=\angle BAC+\angle CFD+\angle DEB+(\angle BCA+\angle BCD+\angle DCF)$$
$$+(\angle ABC+\angle CBD+\angle DBE)+(\angle FDC+\angle CDB+\angle BDE)$$
$$=\angle BAC+\angle CFD+\angle DEB+6R.$$

On the other hand, because sum of the internal angles of a triangle is equal to or less than twice of right angle and sum of the internal angles of triangle ABC≡triangle DCB is $2R-\delta$,

$\angle BAC+\angle CFD+\angle DEB+\angle 6R \leqq 2(2R-\delta)+2\times 2R$ holds.

From above arguments we understand that

$\angle BAC+\angle CFD+\angle DEB \leqq 2R-2\delta$

This is nothing but sum of the internal angles of triangle AEF. I.e., sum of the internal angles of triangle AEF is $2R-2\delta$. Further, repeating the same process on triangle AEF gives a triangle whose sum of the internal angles is $2R-4\delta$. Therefore, repeating the process for n times would give a triangle whose sum of the internal angles is $2R-2n\delta$.

However, because $\delta > 0$, once we choose a large enough n, it becomes possible to have $2R-2n\delta < 0$. On the contrary, because it is sum of the internal angles of a triangle, the value must be positive. This is a contradiction. This was caused because we first assumed that sum of the internal angles of triangle ABC is always smaller than twice of right angle. Therefore, sum of the internal angles of any triangle is twice of right angle.

This makes the end of proof, but it is clear that this proof does not hold. In other words, it is clear that it is not trivial that a straight line passing through point crosses two half lines made by extention of AB and AC. This is the significant mistake in this proof. However, we think that how we can educationally use this mistake itself as a vivid historical teaching material for high school mathematics instead of just discarding it as a mistake.

5. Relationship with Mathematical Activity of Government Guidelines for Education in Japan

In government guidelines for education in Japan, mathematical acitivity is one of the key elements. There, progress of classroom lecture is described by spiral, but when we invesitgate diagram are, the four steps of geometry by Shoukichi Iyanaga may be useful. The four steps is given below: (1) Intuition Step…Step of elementary school education, (2) Local Step (Thales Step)…Step of junior highschool education, (3) Systematic Proof Step (Euclid Step)…Step of high school education, (4) Axiomic Step (Hilbert Step)…Step of university education. "Elements" by Eulid, described in this papar, perfectly corresponds to the step of high school education pointed out by Shoukichi Iyanaga. He describes that (3) experience in high school will lead to (4) axiomic step in university. Also in such a context, the history of the proof of the fifth axiom in Elements‘ by Euclid, described in this paper, can be a valid teaching material for high school education.

6. Conclusion

In this paper, we showed the history of mistakes in the proof of the fifth axiom. Though they are wrong as a result, proofs created by beating the brains contain educational value which is much more than just considering it as a mistake. Seeing this from an educational view point, we think is contains very interesting way of thinking for proof. We hope that teachers pay more attention to the fact that highly interesting matter is hidden in wrong proof itself. In school mathematics, rigorous proof is extremely important. At the same time, we think that intuition is also important. We think that high school mathematics is, in the end, is intuition geometry.

Therefore, it is educationally important to higly esteem inutition while considering connetion of both. We feel that school education was designed with emphasis on logic, but we think that we also should pay more attention on intuition. In such a sense, this paper may be maverick from traditional main stream math education, but on the other hand, we think this is the reality and current status. We believe that there is a possibility that we develop a teaching material to realize mathematical activity in Japan.

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