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Forces Hindering Development of Mathematical Problem Solving among School Children

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Abstract: Problem solving has been a consistent objective of mathematics education for over a century and yet improving mathematical problem solving among children remains an elusive goal (Sriraman & English, 2010). Current research indicates presence of misconceptions about the nature of mathematical problem solving among both teachers and learners. In our presentation, we plan to offer an overview of current research on teaching and learning mathematical problem solving to highlight some of these misconceptions. We will draw from our longitudinal research project that examines the development of mathematical thinking among approximately eighty 7th through 9th grade children from 14 different urban schools to illustrate the direct link between instructional approaches used in classrooms and students' cognition, dispositions towards and during mathematical solving.

Key words: mathematical problem solving, mathematical thinking, mathematics instruction, cognition

1. What is Mathematical Problem Solving?

When two people talk about mathematical problem solving, they may not be talking about the same thing (Wilson, Fernandez, & Hadaway, 1993, p. 1).

Mathematical problem solving has commonly been described in many ways, including "what one does when one does not know what to do", "thinking critically about something that needs solving", "searching for best solution", "working on problems that are complex", "working on ill-defined, open-ended, and real-world problems", to list a few. These descriptions, collectively, suggest that problem solving is an activity during which the problem solver aims to find an appropriate way to cross a gap from a problem to a solution space (Flower & Hayes, 1981). The person has no readily available procedure for finding the solution and he or she must make an effort to find it. Since the goals of the problem may or may not be clear, the problem solver needs to first represent the problem and then monitor solution process. As such, problem solving demands that the problem solver engage in a variety of cognitive and metacognitive actions, involving both routine and non-routine knowledge and skills (Lester & Kehle, 2003).

Certainly, whether a task is perceived as a problem depends largely on individuals' background knowledge and their experience with the type of task under study. What might be assumed to be a problem for one person

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could easily be perceived as an exercise by another individual. To address this, Lesh and Zawojewski (2007) argued that a task becomes a problem when the problem solver needs to develop a more productive way of thinking about the given situation (p. 782). At the heart of this description is the presence of: (1) a particular relationship between the individual (problem solver) and the task (problem) and (2) reflective personal engagement (and investment) in the process: acknowledging the presence of an obstacle, recognizing the need to seek alternative approaches for solving the problem, and a willingness to do so. Consider, for a moment, the three tasks presented in the table below:

Task I	Joe gave Nick and Tom 52 dollars. He gave Nick four times as much money as he gave Tom. How much money did Joe give each one?
Task II	Joe has 52 dollars and wishes to invest his money in two different business ventures initiated by his two friends, Tom and Nick. He would like to earn the greatest amount of interest on his investment. How much should he invest in each business?
Task III	Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. Lastly, Tom gives Joe and Nick as much money as each of them then has. If at the end Tom has 16 dollars and each of the others has 8 dollars, how much money did each have at the beginning?

There is evidence that tasks of the same type as Task I are used as the primary vehicle for developing mathematical problem solving skills among students internationally. However, the remaining two types are rarely utilized in mathematics education of school children (English & Sriraman, 2010). Certainly, teachers make decisions about what tasks to use in class based on their perception of the needs of their students and what they wish students to attain from the tasks. Coupled with teachers' concerns for the type of skills and mathematical tools that might be accessible or necessary, tasks are used in the service of helping children learn specific skills, procedures and techniques, often in an isolated manner. Research has also established that this approach to teaching and task usage is not an effective way of supporting children's development of problem solving skills. Wilson (2003) argued that reducing children's mathematical experiences to solving only a narrow range of tasks as a means to practice skills only teaches students "what to think". Mathematical problem solving, he argued, is about knowing "how to think!" Building on Wilson's argument, we further suggest that this practice can hinder the development of mathematical problem solving skills among children. In doing so, we will first present an overview of the mathematical behaviors of two students toward the same tasks presented above. Drawing from their examples, we will then highlight differences between the activity of solving problems, as done in schools, and problem solving.

2. Getting to Know Troy and Andrea

> If it gets hard you are probably over-thinking it because all the stuff can be solved using what is in the book. (Troy, 7th grade student)

> I like math because there is always an equation I can use to solve a problem. No matter what the problem is, we first set up the equation then it just follows. (Andrea, 9th grade student)

The above statements were made by Troy and Andrea, two successful students as indicated by their high grade point average and performance on standardized examinations. At the time of the interview, Troy and Andrea were in 7th and 9th grades, respectively. They both enjoyed mathematics and appreciated it because "it" challenged them. They both believed they were good in mathematics because they could solve problems quickly and usually without any help from their teachers. We presented the three tasks discussed earlier to each of the children and encouraged them to solve them using any method they deemed appropriate.

2.1 Troy

Upon encountering the first task, Troy immediately divided 52 by 3 to obtain 13, explaining that since the problem was about sharing, and involved three people he had to first do division. Multiplying 14 by 4 to determine the amount of money given to Nick, he recognized $64 was more than what he wanted and asked if he was doing the problem correctly. Rather than answering his question, we asked what he thought. He returned to the problem, set up a table of values, starting with 13 (he knew he needed less than $14 as an initial value) and using guess and check strategy. He found, after approximately 6 trials, the amount of money that Joe had given to each friend. He stated he enjoyed solving the problem and was confident in his answer.

When we changed the parameters of the task by making minor variations to the amount of money shared (what if Joe had give 125 dollars to the two friends), or how the amounts given to the two friends were related (what if the amount he gave Nick was 7 dollars more than 5 times the amount he gave Tom), he managed those tasks equally as easily, using the same methods he had used before. Interestingly, he perceived each extension we posed as a new task. He did not try to use the amounts he had computed in previous cases as a starting point for making educated guesses each time. Also, he used his initial strategy of dividing by three, even when the relationships presented were additive as opposed to multiplicative and despite the fact that the strategy had not helped him reach his answers before. He mentioned that he enjoyed the problem because it made him think but was dissatisfied that it took him so long to solve it.

When we presented him with the second task, Troy responded that he felt the problem was not mathematical since it did not include information about how the money must be distributed among the friends. Even when we insisted if he could describe some of the issues he would consider when deciding how to share the amount among friends, he thought there were way too many factors to consider and none of which he believed was mathematical.

In tackling the third problem, Troy used the same strategies of setting up a table of values, and testing different amounts of money exchanged at each transaction stage by each of the friends. After approximately 6 minutes of testing for different numbers, due to a lack of progress, he asked if the problem even had a solution. He finally gave up working on the problem, arguing that it really was not a mathematical problem because it did not ask specifically what he needed to find. “If it takes this long to do it, then it probably does not have an answer or I may be over-thinking it; solving mathematical problems should not take this long,” he concluded.

2.2 Andrea

After reading the first task, Andrea stated that he had done this kind of problem *a million times before*. His immediate approach was to first represent the amounts given to friends symbolically (writing T and 4T). After four trials of guess and check he arrived at the amounts given to friends within 50 seconds. He initiated testing the values he had derived. Similar to Troy, when we presented him with extensions of the task, Andrea had no difficulty deriving answers. Indeed, he stated that regardless of how difficult we tried to make the problem (by either increasing the number of friends or changing the relationships among the amounts) he could probably solve it. He was indeed correct, despite never using algebra or knowledge of solving equations.

In reacting to the second task, Andrea stated that he needed a lot more information about each of the business ventures before he could answer the question. He too stated that he did not believe the task was mathematical since it did not involve specific numbers or relationships that could be translated into an equation to be solved.

After reviewing the statement of the third problem, Andrea expressed concerns about whether all information he needed was available since the problem did not start with a certain amount. We reassured him that all information he needed was present. This motivated him to go back to the problem and after about 30 seconds he concluded that he believed Nick and Joe each had 2 and Tom 4 dollars at the beginning, since their amounts of money were doubled at the end of each transaction. He was confident that he had solved the problem. When we asked if he could prove that his response was right, he modeled the transactions with the amounts he had derived and recognized that his answers were incorrect. However, in his second attempt at solving the problem he abandoned the strategy of working backward, returning to the guess and check method. After working on the task for another 4 minutes, he remarked that he was certain there was an equation he could use to solve the question but he did not know what it was. "There is always an equation; there is always a way of writing an equation and solving problems," he concluded.

We shared Troy's and Andrea's cases here not because they are unique but because they are representative of how a majority of the children in our study think and talk about mathematics, and solve or refuse to solve problems. Their behaviors are symptomatic of what a large number of children in schools come to believe about mathematics, and learn to assume about mathematical problems. Troy's and Andrea's mathematical behaviors and their reactions to tasks they encountered are illustrative of the unintended (and yet unpleasant) outcomes of repeated exposure to the experience of practiced *solving of problems*, as it is frequently done in schools, referenced as *problem solving*. Children learn to manipulate symbols and numbers without reflecting on the context, fail to seek connections among ideas or concepts they already know and what the problem asks for, neglect to draw knowledge from what they may have already done to organize and/or generalize solutions and solution strategies, come to believe if problems are not solved quickly they should be abandoned, and develop intolerance for ambiguity and a need for immediate closure. These types of learned behaviors and dispositions hinder children's mathematical problem solving ability and their desire to engage in the process.

3. Solving Problems vs. Problem Solving

Solving problems, as it is utilized in school mathematics curriculum and instruction, involves learners working on well-structured and well-defined tasks. Frequently used in textbooks and presented to students as chapter assignments or homework activities, the goal of solving problems is to help students see how certain algorithms or procedures might be used efficiently. Indeed, the classroom practice of solving problems requires the application of a set of rules for manipulating equations or figures; there is an optimal solution path (illustrated in the textbook, modeled by the teacher) and learners are asked to follow a pre-defined logical system. Whether the problems involve taking a single step or multiple steps for launching answers, their intent is to evoke particular actions and to standardize learners' mathematical performances. Therefore, the scope of the problems used is **narrow**. The students' mastery of skills or lack thereof, is often determined based on the speed at which they can apply these algorithms and procedures in similar contexts.

Due to the extreme care that teachers take, both in sequencing the tasks and in presenting them to students, solving problems is **safe**. Teachers shepherd students through the process by limiting the range of decisions children must make in order to assure desired outcomes are obtained. By modeling how problems should be solved via showing examples, and providing a structure for children's thinking, teachers unintentionally limit students' intellectual space for inquiry. Indeed, the activity of *solving problems* is a **rehearsal** of a script that is expected to be produced.

In contrast with solving problems, *problem solving* relies heavily on the problem solvers' in-the-moment decision making. While the problem solver may have a large (or small) amount of mathematical tools at their disposal, knowledge about how to select, flexibly from the collection, those tools most appropriate to the context under study becomes a pivotal part of the process. Flexibility in the use of tools implies that the individual is able to determine their effectiveness in context. As such, the activity of problem solving involves a range of cognitive and metacognitive actions, starting with understanding the problem, devising a plan for solving it, monitoring progress, and testing and verifying approaches and solutions. In doing so, as Schoenfeld (1992) argued, problem solving involves drawing from both domain specific knowledge, consisting of rules, procedures and algorithms, and ways in which that knowledge is structured and organized. Drawing from this network, the problem solver must navigate the problem, strategize, assess, and consider alternatives. Problem solving demands ***improvising.***

In problem solving, since the goals of a problem may not be clear, and a direct path or preferred technique for launching answers may not be immediately accessible, the problem solver is vulnerable to failure. As the problem solver carries out plans, tests ideas, and struggles to solve the problem, there is potential for her/him to make new discoveries, notice unexpected patterns, and recognize the need to develop new mathematical ideas s/he may not have experienced before. Therefore, the scope of the task is **wide**. These new observations and discoveries, while exhilarating, may not move the problem solver towards a solution. Therefore, the activity of mathematical problem solving can be dangerous and **risky**, not to mention time and labor intensive. The amount and quality of learning that might arise from problem solving by different individuals cannot always be standardized. This contradicts the very essence of schooling as it has traditionally been practiced and perceived. Mathematical problem solving is not safe since it is not prescriptive.

There is evidence that the activity of solving problems constitutes a large portion of mathematics instruction in the United States (TIMSS, 2003). There is also ample evidence that solving problems, when treated as an end in itself (Schoenfeld, 1992), does not improve problem solving skills among learners. Transfer of knowledge from the domain of practiced solving to problem solving is not easily achieved (Cai, 2010). Students, even those who possess a substantial amount of sophisticated mathematical tools and vast experience with solving problems, perform poorly when confronted with tasks that demand intellectual improvising.

Over three decades ago, Begle (1979) argued that the view that mathematics is first developed and then applied to routine problems functions as a major obstacle to the development of problem solving skills. Thirty years of research on mathematical problem solving following Begle’s work indicate that delaying children’s authentic mathematical experiences until skills are established is not a fruitful approach for enhancing mathematical problem solving among them. Yet, there is evidence that mathematics instruction continues to treat teaching problem solving in the same fashion (Sriraman & English, 2010). Reports indicate that common approaches to teaching mathematical problem solving consist of teachers using problems they find relevant to concepts or skills they introduce in class without making an effort to help children seek connections among concepts or problem structures (Lesh & Zawojewski, 2007). As Wilson (1998) explained, such sequential and structured experiences counter the essence of mathematical problem solving and ultimately mathematical thinking as it is exercised outside classrooms and in mathematics community.

4. Dispositions and Cognition: Impact of Repeated “Practiced Solving”

Previous research has identified a number of factors that impede children’s problem solving performance, consisting of both affective and cognitive elements, including: lack of persistence (Schneider & Artelt, 2010), inability to monitor progress (Cohors-Fresenborg et al., 2010), immature organizational skills (Stillman & Mevarech, 2010), inability to distinguish between relevant and irrelevant data (Ge & Land, 2003), and lack of strategic thinking (Malloy & Jones, 1998). There is also evidence that students' sense of efficacy and confidence (Williams et al., 1988) and beliefs about the nature of mathematics impact significantly their mathematical problem solving performance. In synthesizing a large body of literature, Cai (2010) reported that knowledge, beliefs, views about the nature of mathematics and thinking processes are conceptually intertwined (p. 253). That is, what students come to believe about the nature of the subject matter, how the subject is developed, and what is valued in the study of the discipline impacts both their cognitive and metacognitive behaviors when encountering problems. These beliefs, dispositions, and cognitive processes are shaped, prominently, by children's school experiences.

We argue that extensive exposure to the activity of solving problems (practiced solving) negatively impacts children's mathematical thinking and their success as mathematical problem solvers by automizing their behaviors, to the extent of minimizing children's need to learn to reason or reflect on concepts, tasks or connections among them. This in turn restricts the domain of conceptual exploration in which children would be willing to engage and, thus, constraints the development of flexible thinking among them. We will elaborate on these points in greater detail in the following sections.

4.1 Automizing Actions

> If I am not getting the right answer it means that I don't have the problem right. (Troy)
>
> I like math because there is always a right way of doing things and if you do it you always get the right answer. (Andrea)

In classrooms, structured, practiced solving occurs frequently after teachers introduce an algorithm/procedure and show examples on the board. Students are then assigned either specific worksheets that they complete in class or sent home with exercises to do (Campbell, 2006). To increase students' efficiency in solving problems, teachers advise them to recognize problem types and recall the specific procedures that apply. They also teach students to look for specific words in the problem statement in order to quickly interpret what the problem might be asking them to do. These practices provide a consistent and standard platform for students' thinking and performance and automize their actions. Naturally, when children's reflection on problem contexts is reduced to recall of specific information, they will have limited need to reason or learn to think flexibly, two crucial skills needed for successful problem solving (Elia et al., 2009). Extended experiences with solving well-structured problems, that reinforce the mastery of application of specific algorithms sequenced according to specific concepts, do not allow children to learn to analyze problem contexts conceptually or differentiate among strategies and tools according to their utility in contexts. This knowledge base is needed in order for children to learn to engage in adoptive and flexible thinking.

As English and Sriraman (2010) concluded: knowing when, where, why, and how to use heuristics, strategies, and metacognitive actions lies at the heart of what it means to understand them (p. 265). Performance, when not grounded in understanding, becomes mechanical, void of personal connection on the part of the problem solver to either the problem context or the tools effective for studying it.

4.2 Dealing with Ambiguity: The Dilemma of Clarity in Instruction

> If you give me the equation I can probably solve this. (Andrea, approaching the third problem)

> I love my teacher because she is clear; if I don’t understand a problem I ask her and she explains it to me. (Troy)

Literature has consistently identified the capacity to deal with irrelevant information (and detail) as a key to successful problem solving (Hembree, 1992). The ability to deal with ambiguity and reasoning under hypothesis becomes particularly critical in dealing with problems that arise in real-life contexts (type II problem) where neither the problem nor the variables involved are clearly defined. For instance, in mathematical modeling contexts the individual needs to define the parameters and variables, decide the type of data needed, find the data, and use as well as analyze the data as a means to make decisions regarding the *most appropriate* solutions to the problem under study. Solutions are judged based on their ability to account for the greatest number of variables, increasing the predicative power of the model. Arguably, the most labor and time intensive part of solving “real” problems is identifying and setting up the problem since it is an ongoing part of the process. Indeed, full understanding of the problem situation may not occur at the initial point of entry to the task and is often deepened as the problem solver further explores conditions and circumstances that can impact it. The skills needed for navigating these types of tasks demand that the individual have a variety of “go-to” techniques and also be able to monitor the constraints and limitations of each of them. Consequently, the ability to organize knowledge becomes crucial in the process. Problem solvers need to draw from what they already know to make sense of what is proposed and resolve ambiguities by creating connections between what is already known and the task at hand. In this context, the quality of the organization of students’ knowledge becomes a far more influential factor for success than the quantity of disconnected knowledge pieces they might possess (Lawson & Chinnappan, 2000).

Developing children’s ability to connect and organize knowledge has not been a consistent focus of mathematics instruction. In fact, as studies have shown, when posing questions, teachers assure that the goals of tasks are clearly stated and explained to children; they remove ambiguities from problems and even outline for children the steps they need to take to deduce answers in order to increase learners’ ability to control the task domain. Although this approach often yields desirable short-lived outcomes (good performances on tests), it can have serious negative long-term impact on children’s ability to think and problem solve. Indeed, when this controlled approach is experienced over a long period of time, it disables children from autonomous and reflective decision-making. Children are taught to search for clues or problem types. If they fail to recognize clues or those they know are not present in the text, they often abandon the problem, assuming it is enigmatic or non-mathematical. Additionally, the expectations that children develop as the result of repeated exposure to solving exercises provide them with a narrow, if not false, framework for judging their own progress on tasks. These were manifested in Troy’s and Andrea’s mathematical behaviors and how they viewed and interacted with tasks.

5. Looking Back, Moving Forward

In recent years there has been substantial debate on whether research has been successful in helping the community better understand how mathematical problem solving might be nurtured and linked to conceptual development of children (English & Sriraman, 2010). While these arguments are certainly healthy in defining pathways for future inquiries that may need to take place, and it might be true that we have not yet, as a community, reached consensus on how to best and most effectively teach mathematical problem solving to all students, we are positioned well in our understanding of what techniques are not successful in fostering the type of intellectual flexibility that mathematical thinking demands.

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